1. (10) Determine if the improper integral

$$\int_0^\infty \frac{x}{1+x^2}\,dx$$

converges or diverges.

Let $u = 1+x^2$, $du = 2x\,dx$

the indefinite integral is $\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(u)$

to evaluate, take

$$\lim_{t\to\infty} \frac{1}{2}\ln(1+x^2)\Big|_0^t$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\ln(1+t^2) - \underbrace{\frac{1}{2}\ln(1)}_{=0}\right)$$

$$= \frac{1}{2}\lim_{t\to\infty}\ln(1+t^2)$$

since $1+t^2 \to \infty$ as $t\to\infty$

$\ln(1+t^2)\to\infty$ as $t\to\infty$

The limit does not exist, so the integral diverges.

2. (12) Determine if the series

$$\sum_{n=2}^{\infty} \frac{\sqrt{2n}}{n^2-1}$$

converges. Mention any test that you might use and verify that it is applicable.

All terms are positive. $\frac{n^{1/2}}{n^2}$ is like $\frac{1}{n^{3/2}}$ which gives a convergent p-series (also with all positive terms) so I will use limit comparison.

$$\lim_{n\to\infty} \frac{\sqrt{2n}/(n^2-1)}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\sqrt{2}\, n^{1/2}\, n^{3/2}}{n^2-1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{2}\, n^2}{n^2-1} = \sqrt{2}$$

Since $0 < \sqrt{2} < \infty$ the limit test shows $\sum_{n=2}^{\infty} \frac{\sqrt{2n}}{n^2-1}$ and $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ have the same convergence properties.

Since the latter is a convergent p-series, the former (given) series converges.

3. (14) The following power series has radius of convergence $R = 7$.

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt[3]{n}\, 7^n}$$

Find the interval of convergence. Mention any test that you might use and verify that it is applicable.

We need to check the endpoints of $(2-7, 2+7)$.

$x = -5$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-7)^n}{\sqrt[3]{n}\, 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

Since $\frac{1}{\sqrt[3]{n}}$ decreases to zero, the Alternating series test shows this series converges.

$x = 9$: $\displaystyle\sum_{n=1}^{\infty} \frac{(7)^n}{\sqrt[3]{n}\, 7^n} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ This is a divergent p-series.

Hence the interval of convergence is $[-5, 9)$.

4. (12) Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{nx^n}{(n+1)!\,2^{n-1}}.$$

Ratio Test

$$|a_{n+1}| = \frac{(n+1)\,|x|^{n+1}}{(n+2)!\;2^n} \qquad |a_n| = \frac{n\,|x|^n}{(n+1)!\;2^{n-1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\,|x|^{n+1}\;(n+1)!\;2^{n-1}}{(n+2)!\;2^n\;(n)\;|x|^n} = \frac{(n+1)\,|x|}{2n\,(n+2)}$$

$$= \frac{|x|(n+1)}{2n^2+4n}$$

$$\lim_{n\to\infty} \frac{|x|\,(n+1)}{2n^2+4n} = 0 \quad \text{independent of the value of } x$$

so the radius of convergence is ∞.

5. (10) Suppose that $f(x)$ is equal to its Tayor series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}(x-3)^n$$

about $a = 3$. What is the 39th derivative $f^{(39)}(3)$? You need not simplify your answer. No partial credit will be given for this problem.

The term $n = 39$ of the Taylor series is

$$\frac{f^{(39)}(3)}{39!}(x-3)^{39}$$ from the definition

and $$\frac{1}{2^{39}}(x-3)^{39}$$ from the equation given here.

Thus $$f^{(39)}(3) = \frac{39!}{2^{39}}.$$

6. (12) Write down the first three non-zero terms of the Taylor series for $\ln(2x+4)$ at $a = 1$.

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$n!$
0	$\ln(2x+4)$	$\ln 6$	1
1	$\frac{1}{x+2}$	$1/3$	1
2	$\frac{-1}{(x+2)^2}$	$-1/9$	2

$$\ln 6 + \frac{1}{3}(x-1) - \frac{1}{18}(x-1)^2$$

7. (12) Express the integral

$$\int 2(2+x)^{-1}\,dx$$

as a MacLaurin series. It suffices to write down the first four non-zero terms. You may assume that the arbitrary constant $C = 0$.

$$\frac{2}{2+x} = \frac{2}{2(1+\frac{x}{2})} = \frac{1}{1+\frac{x}{2}} = \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$\int \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n dx = \sum_{n=0}^{\infty}\int \left(-\frac{1}{2}\right)^n x^n\,dx$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \frac{x^{n+1}}{n+1} \qquad (+0)$$

first four terms:

$$x - \frac{x^2}{4} + \frac{x^3}{12} - \frac{x^4}{32}$$

8. (18) For each of the following statements, fill in the blank with the letters **T** or **F** depending on whether the statement is true or false. You do not need to show your work and no partial credit will be given on this problem.

(a) The sequence $\left\{ \left(\frac{\pi}{3}\right)^n \right\}$ converges.

$\frac{\pi}{3} > 1$ so $\to \infty$

ANS: F

(b) The series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n)$ exists.

$\lim_{n\to\infty}$ (partial sums) needs to be finite

(def. of convergence for series)

ANS: T

(c) The series $.9 + .99 + .999 + \cdots$ converges to 1.

the sequence .9, .99, .999, ...
converges to 1, but that means
this series diverges by the Test for
Divergence.

ANS: F

(d) If $\sum_{n=1}^{\infty} a_n$ is a divergent series, then $\sum_{n=1}^{\infty} |a_n|$ is a divergent series.

This is equivalent to the statement
"if $\sum a_n$ converges absolutely, then it converges"
(contrapositive)

ANS: T

(e) $\lim_{n\to\infty}(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}) = 1$.

geometric $\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n$

since $\frac{1}{2}<1$, conv. to $\frac{1/2}{1-1/2} = 1$

ANS: T

(f) If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

exactly the statement of the (Direct) comparison test.

ANS: T