Math 8, Winter 2005

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Improper Integrals

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$$\int_a^b f(x)\, dx$$

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These are called *improper integrals* and require careful handling.

Standard example: $f(x) = \frac{1}{x}$

We know:

$$\int_a^b \frac{1}{x}\,dx = \ln(b) - \ln(a)$$

Improper Integrals

- Letting $a = 1$,$b = t$ we have

$$A(t) = \int_1^t \frac{1}{x}\, dx = \ln(t) - \ln(1) = \ln(t)$$

- $$\lim_{t \to \infty} A(t) = \infty$$

- In other words, as t grows, the area under this curve tends to ∞.

Improper Integrals

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- Consider

$$\begin{aligned} A(t) &= \int_1^t \frac{1}{x^2}\,dx \\ &= -\frac{1}{x}\Big|_1^t \\ &= -\frac{1}{t} + 1 \end{aligned}$$

- As $t \to \infty$, $A(t) \to 1$.

Improper Integrals

So, we define the value of an integral of the form

$$\int_a^{\infty} f(x)\, dx$$

to be

$$\lim_{t\to\infty} \int_a^{t} f(x)\, dx$$

If the limit tends to $\pm\infty$, we say the integral *diverges*. If, instead, it tends to a finite value, we say the integral *converges* to that value. Similarly,

we define the value of an integral of the form

$$\int_{-\infty}^{b} f(x)\, dx$$

to be

$$\lim_{t\to-\infty} \int_t^{b} f(x)\, dx$$

Improper Integrals

And, if both bounds are infinite:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

Examples:

- $$\int_{1}^{\infty} \frac{1}{(3x+1)^2}\,dx$$
- $$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$$
- $$\int_{0}^{\infty} \cos^2(\theta)\,d\theta$$

Improper Integrals

Another type of improper integral: integrals where the integrands have discontinuities. Example:

$$\int_0^3 \frac{dx}{x-1}\,dx = \lim_{t\to 1}\left(\int_0^t \frac{dx}{x-1}\,dx + \int_t^3 \frac{dx}{x-1}\,dx\right)$$