Math 8, Winter 2005

Scott Pauls

Dartmouth College, Department of Mathematics

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Numerical methods

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Examples:

$$\int e^{-x^2}\, dx$$

$$\int \sqrt{1+x^3}\, dx$$

Numerical methods

In practice, we use numerical approcimations to determine the (approximate) values of integrals. We already know one technique: Riemann sums.

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- Right endpoints:

$$\int_a^b f(x)\, dx \sim R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

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We have a further refinements. First, the **trapezoidal rule** where we average the left and right endpoint approximations:

$$\int_a^b f(x)\,dx \sim T_n = \frac{\Delta x}{2}(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n))$$

Examples

Examples: Use the various methods to estimate the following integrals.

- $$\int_0^1 x^3 \, dx$$
- $$\int_0^1 e^x \, dx$$

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- An upper bound on this quantity is called an **error estimate**.
- for the midpoint rule, the following estimate is known:

$$E_M(n) \leq \frac{K(b-a)^3}{12n^2}$$

where K is an upper bound for $|f''(x)|$ for $a \leq x \leq b$