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Local extrema

In our analysis of the gradient, we deduced that local maxima and minima appear as critical points:

The *local extrema* of a function $f(x, y)$ are characterized by the equation

$$\nabla f = 0$$

At these points, the tangent plane to the surface $x = f(x, y)$ is horizontal.

Classifying extrema

Examples:

- **Local minimum**: $f(x,y) = x^2 + y^2$, $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$
- **Local maximum**: $f(x,y) = -x^2 - y^2$, $f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 0$
- **Saddle point**: $f(x,y) = x^2 - y^2$, $f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$
- **Saddle point**: $f(x,y) = xy$, $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = 1$

Classifying extrema

As in one variable calculus, we have a second derivative test that tells us the nature of the critical points:

Second Derivative Test: Let

$$D = f_{xx}f_{yy} - f_{xy}^2$$

and (x_0, y_0) a critical point of f. Then,

1. If $D(x_0, y_0) > 0$ and $f_{xx} > 0$ then the (x_0, y_0) is a *local minimum*
2. If $D(x_0, y_0) > 0$ and $f_{xx} < 0$ then the (x_0, y_0) is a *local maximum*
3. If $D(x_0, y_0) < 0$ then the critical point is a *saddle point*
4. If $D(x_0, y_0) = 0$ then the test is *inconclusive*

Examples

- $f(x, y) = x^2 - 2xy + y^4$
- $f(x, y) = exp(-x^2 - y^2)$

Absolute extrema

If D is a closed and bounded region in $\mathbb{R}^2$ we can find the absolute maximum and minimum of a function $f(x, y)$ on that set using the folowing procedure:

1. First find all critical points of f inside the set D and plug them into f.
2. Parameterize the boundary of D and
   (a) Restrict the function f to the boundary using this parameterization
   (b) Find the abosulute maximum and minimum of f along the boundary
3. Pick the largest and smallest values from the list

Example: $f(x, y) = x^2 - 2xy + y^4$ with D the square of side length one with one corner at $(0, 0)$ and the opposite corner at $(1, 1)$.