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Linear Approximations

Just like we used the tangent line to approximate a function of one variable, we can use the tangent plane to approximate a function of two variables: Given a function $f(x,y)$ and its tangent plane at $(x_0, y_0, f(x_0, y_0))$, $z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$, the tangent plane is a good approximation of the function near (x_0, y_0). i.e.

$$f(x,y) \approx L(x,y)$$

where

$$L(x,y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

Examples

- Find the tangent plane to $f(x, y) = 2x^2 + 3xy$ at $(1, 1)$
- Approximate $f(1.1, 0.9)$ where $f(x, y) = 2x^2 + 3xy$ near $(1, 1)$

When thinking about differentiability, we would like that a function f be well approximated by its tangent planes.

Differentiability

If $z = f(x, y)$, then f is *differentiable* at (a, b) if $\Delta z = z - f(a, b)$ can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(x, y) \to (a, b)$

If f_x and f_y exist near (a, b) and are continuous then f is differentiable at (a, b)

Chain rule

Let $z = f(x, y)$

- If $x = g(t), y = h(t)$, then

$$\frac{d}{dt} f(g(t), h(t)) = f_x(g(t), h(t))g'(t) + f_y(g(t), h(t))h'(t)$$

- If $x = g(s, t), y = h(s, t)$ then

$$\frac{\partial z}{\partial s} = z_x x_s + z_y y_s$$

  and

$$\frac{\partial z}{\partial t} = z_x x_t + z_y y_t$$

- General case: follow the tree diagram to get the correct derivative

Examples

- $f(x, y) = x^2y + xy^2$, $x = 2 + t$, $y = t^3$
- Same f, $x = st$, $y = s^2 + t^3$
- $f(x, y, z) = x^2 + y^3 + z^4$, $x = \ln(s)$, $y = st^2$, $z = t^3 + st$