Math 8, winter 2005

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Volume

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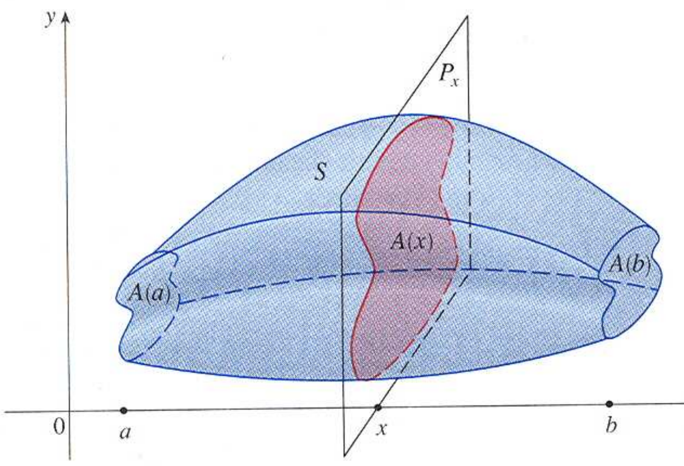

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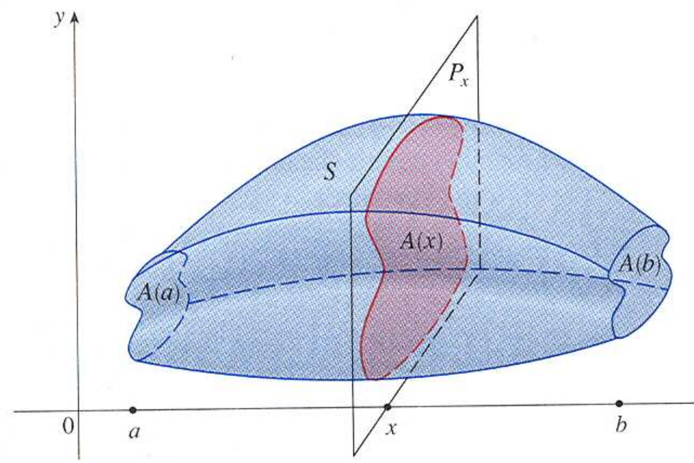
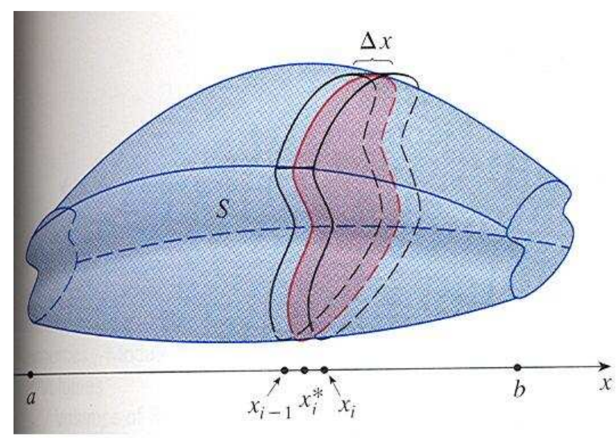


- We understand how to compute area, so let's slice the solid into 2-dimensional sheets. Calculate the area for that slice, $A(x)$.
- Thicken the slice to a slab of width Δx, then the volume of the slab is *approximately* $A(x)\Delta x$.



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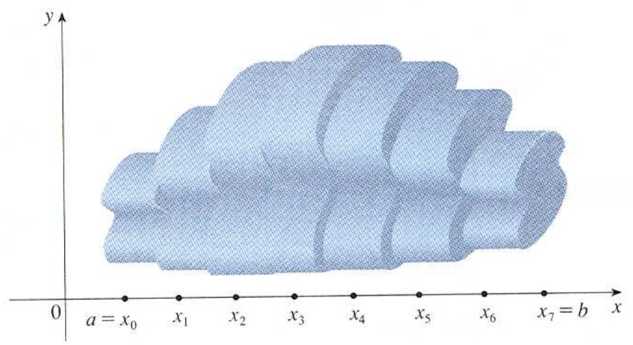
- Sampling many different points in $[a, b]$, we generate an approximation of the volume of the solid:

$$V \sim \sum_{i=1}^{n} A(x)\Delta x$$



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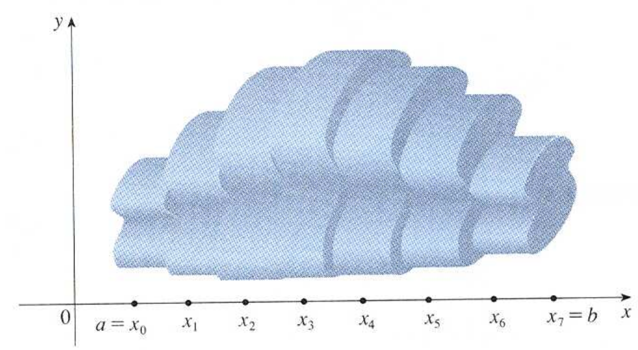

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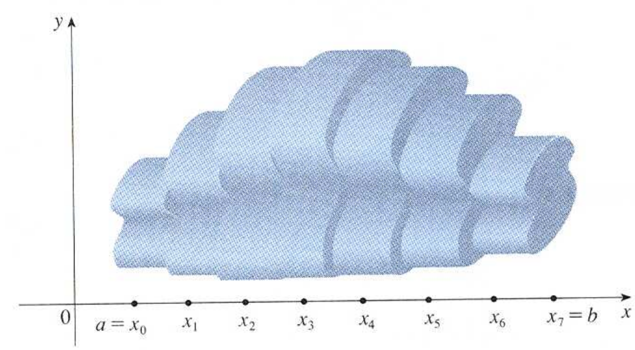

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- Difficulty: Compute $A(x)$.

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- Same problem but rotate about the y-axis.

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- Find the volume of a solid whose base S is the parabolic region $\{(x, y)|x^2 \leq y \leq 1\}$ and whose cross-sections perpendicular to the y-axis are equilateral triangles.

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$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

- Integrate both sides to get:

$$f(g(x)) = \int_a^b \frac{d}{dx} f(g(x))\, dx = \int_a^b f'(g(x))g'(x)\, dx$$

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$$\begin{aligned} f(x)g(x)|_a^b &= \int_a^b \frac{d}{dx}f(x)g(x)\;dx \\ &= \int_a^b f'(x)g(x)\;dx + \int_a^b f(x)g'(x)\;dx \end{aligned}$$

Integration by parts

- Rearrange terms to get the *integration by parts* formula:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)\,dx$$

or, letting $u = f(x)$, $v = g(x)$,

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- Key is simplicity of resulting integral: pick u so that du is simpler, pick dv so that v is *at least* not much worse than dv.

Integration by parts

- $$\int xe^x \, dx$$
- $$\int x^2 \sin(x) \, dx$$
- $$\int \arctan(x) \, dx$$
- $$\int \ln(x) \, dx$$
- $$\int e^x \sin(x) \, dx$$

Rules of thumb

- Choices for u: polynomimals, arc-trig functions, logarithms, $\sin(x)$, $\cos(x)$
- Choices for dv: e^x, $\sin(x)$, $\cos(x)$, polynommials