Math 8, Winter 2005

Scott Pauls

Dartmouth College, Department of Mathematics

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Power series representations

What about a function like

$$f(x) = \sin(x)$$

Idea:

- Goal: a power series representation about a
- The m^{th} partial sum of a series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

  is a polynomial of degree m:

$$s_m = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_m(x-a)^m$$

- Find polynomials that closely match the function, $f(x)$, near $x = a$.

$m = 0$ and $m = 1$

$m = 0$

$s_0 = c_0$ so simply pick $c_0 = f(a)$.

$m = 1$

$s_1 = c_0 + c_1(x - a) = f(a) + c_1(x - a)$

Find c_1 so that $f(x) - (f(a) + c_1(x - a))$ is as small as possible near $x = a$:

$$f(x) - (f(a) + c_1(x - a)) = (f(x) - f(a)) + c_1(x - a)$$

Dividing by $(x - a)$ gives:

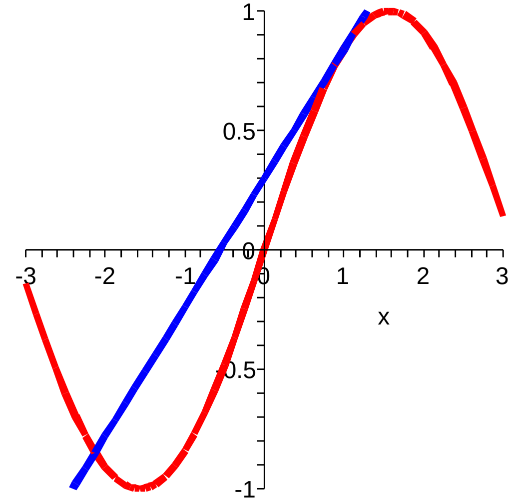
$$\frac{(f(x) - f(a))}{x - a} + c_1 \sim f'(a) + c_1$$

when x is close to a. So pick $c_1 = f'(a)$.

$m = 1$ geometrically

Using these values, what is $s_1 = f(a) + f'(a)(x - a)$?

$m = 1$ geometrically



Using these values, what is $s_1 = f(a) + f'(a)(x - a)$?

It is just the tangent line to f at $x = a$!

Polynomial approximations

If we continue this process, each time we add a term containing higher derivatives of f at $x = a$:

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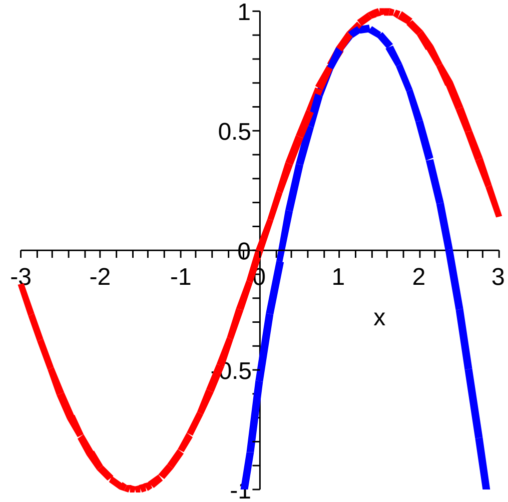
$\boxed{m = 2}$

$$s_2 = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$



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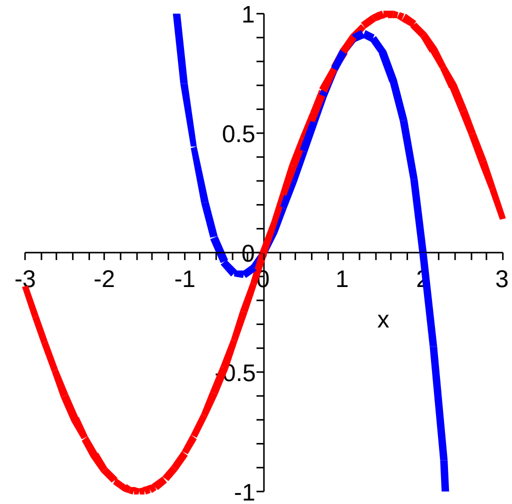

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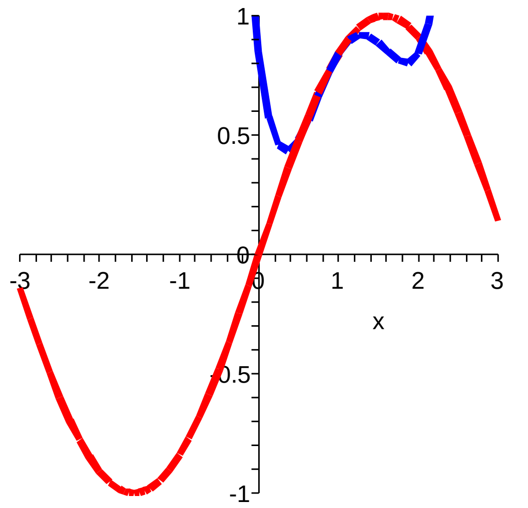

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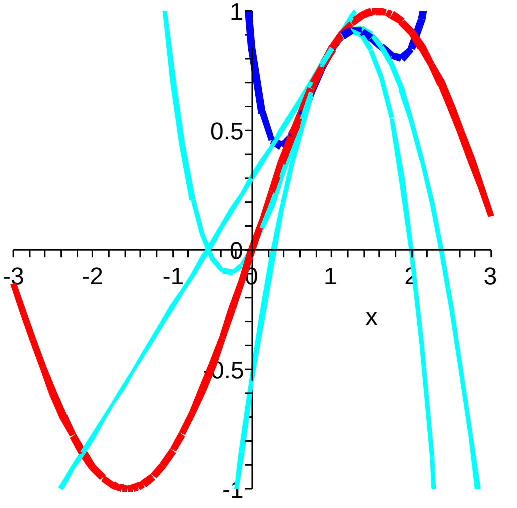

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Taylor series

The polynomials $\{s_1, s_2, s_3, \dots\}$ are called *Taylor polynomials* and are the best order n polynomial approximations of $f(x)$ near $x = a$. This leads us to define:

Given a function, $f(x)$, we define the *Taylor series* of f about $x = a$ by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If $a = 0$, this series is also called the *Maclaurin series* of f

Taylor's theorem

Theorem: Within its radius of convergence, the value of the Taylor series equals the value of the function. In other words, if the radius of convergence of the Taylor series for f is R then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for $|x-a| < R$.

Computing Taylor series

Example: $f(x) = \sin(x)$, $a = 0$

Compute derivatives:

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n	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{1}{n!}f^{(n)}(a)$
0	$\sin(x)$	$\sin(0) = 0$	0
1	$\cos(x)$	1	1
2	$-\sin(x)$	0	0
3	$-\cos(x)$	-1	$-\frac{1}{3!}$
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So the first few terms of the Taylor series are:

$$\sin(x) = 0 + x + 0x^2 - \frac{1}{3!}x^3 + \ldots$$

Maclaurin series for $\sin(x)$

In fact, since the derivatives repeat, we can conclude that

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

or, using summation notation:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Check: $R = \infty$.

Exercises:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Why is this useful?

Recall: we cannot evaluate the integral

$$\int_0^1 e^{-x^2}\,dx$$

using elementary means. But, we can evaluate the integral if we replace the integrand with power series representation:

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,dx \\ &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\Big|_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}\end{aligned}$$

Approximation of the integral

So,

$$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$$

We can use the partial sums to approximate the value:

m	s_m
1	0.6666666667
2	0.7666666667
3	0.7428571429
4	0.7474867725
5	0.7467291967
6	0.7468360343
7	0.7468228068