TAYLOR SERIES WORKSHEET

APRIL 8, 2019

1. Let $f(x) = \sin(x)$. In this exercise you will show that the Taylor series of $\sin(x)$ is $\sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$.

   (a) Compute the following derivatives.

   $$f(x) = \qquad\qquad f(0) =$$

   $$f'(x) = \qquad\qquad f'(0) =$$

   $$f''(x) = \qquad\qquad f''(0) =$$

   $$f'''(x) = \qquad\qquad f'''(0) =$$

   $$f^{(4)}(x) = \qquad\qquad f^{(4)}(0) =$$

   (b) Based on your computations above, find formulas for the even- and odd order derivatives at 0.

   $$f^{(2m)}(0) = \_\_\_\_\_\_\_\_ \qquad\qquad f^{(2m+1)}(0) = \_\_\_\_\_\_\_\_$$

   (c) Using the previous steps, compute the Maclaurin series for $\sin(x)$. Write out the first 5 terms of the series.

   (d) Compute the interval of convergence of the Maclaurin series for $\sin(x)$.

2. As before, let $f(x) = \sin(x)$. In this exercise you will show that the Maclaurin series for $\sin(x)$ converges to $\sin(x)$, that is, $\sin(x) = \sum_{m=0}^{\infty}(-1)^m \frac{x^{2m+1}}{(2m+1)!}$ for all real x.

Let $p_n(x)$ be the n^{th} Maclaurin polynomial and $R_n(x) = f(x) - p_n(x)$ be the n^{th} remainder.

(a) Fix a real number x. What does Taylor's Theorem with Remainder say about the remainder $R_n(x)$?

(b) Find a bound on $f^{(n+1}(c)$ valid for any c between 0 and x and any nonnegative integer n.

(c) Using the previous steps, show that $R_n(x) \to 0$ as $n \to \infty$.