PROPERTIES OF POWER SERIES HANDOUT

APRIL 5, 2019

Theorem (Properties of power series)**.** Suppose that

$$\sum_{n=0}^{\infty} c_n x^n \text{ converges to } f(x) \text{ for all } x \text{ in an interval } I_1$$

and

$$\sum_{n=0}^{\infty} d_n x^n \text{ converges to } g(x) \text{ for all } x \text{ in an interval } I_2.$$

Then:

(a) $\displaystyle\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f(x) \pm g(x)$ for x values in the overlap of I_1 and I_2.

(b) $\displaystyle\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ for x values in I_1 (where b is a fixed value and $m \geq 0$).

(c) $\displaystyle\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ as long as bx^m is in I_1 (where b is a fixed value and $m \geq 0$).

(d) $\displaystyle\left(\sum_{n=0}^{\infty} c_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} d_n x^n\right)$ converges to $f(x) \cdot g(x)$ for x in the overlap of I_1 and I_2. (**Warning:** we cannot multiply series by just multiplying the corresponding terms! We have to distribute.)

(e) $\displaystyle f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ for x values in I_1. (Take the derivative of each term.)

(f) $\displaystyle\int f(x)\, dx = C + \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}$ for x values in I_1. (Take the antiderivative of each term.)

Parts (e) and (f) are called "term-by-term" differentiation and integration.

Note: Parts (a), (d), (e), and (f) still hold if the center of the power series is nonzero, just replace x with $(x - a)$. For parts (a) and (d), both series must have the same center.

Exercises:

1. (a) Use partial fractions to find a power series for $\dfrac{4}{(x-3)(x+1)}$.

   (b) What is the interval of convergence of the series?

2. Find the first 3 terms (constant, x, x^2) of the series obtained by multiplying

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n}} \quad \text{and} \quad \sum_{n=0}^{\infty} n2^n x^n.$$

   Hint: Write out the first few terms of each series and FOIL/distribute. You need at least 3 terms of each. (Why?)

3. (a) Find a power series for $\ln(1+x)$ centered at 0. (*Hint*: $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$.)

   (b) Find a power series for $\ln(x)$ centered at 1.

4. (a) Find a power series for $\arctan(x)$ centered at 0.

   (b) Show that the power series you found in the previous part converges at $x = 1$.

   (c) Evaluate the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.