CALCULUS OF VECTOR-VALUED FUNCTIONS HANDOUT

APRIL 29, 2019

Theorem (Differentiating Vector-Valued Functions)**.** Let f, g, and h be differentiable functions of t.

(i) If $\vec{r}(t) = \langle f(t), g(t)\rangle$, then $\vec{r}'(t) = \langle f'(t), g'(t)\rangle$.
(ii) If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$.

Theorem (Properties Of Differentiation Of Vector-Valued Functions)**.** Let $\vec{r}$ and $\vec{u}$ be differentiable vector-valued functions, let f be a differentiable real-valued function, and let c be a scalar.

(i) $\dfrac{d}{dt}\, c\vec{r}(t) = c\vec{r}'(t)$ (scalar multiplication)
(ii) $\dfrac{d}{dt}[\vec{r}(t) \pm \vec{u}(t)] = \vec{r}'(t) + \vec{u}'(t)$ (sum and difference)
(iii) $\dfrac{d}{dt}[f(t)\vec{r}(t)] = f'(t)\vec{r}(t) + f(t)\vec{r}'(t)$ (rescaling)
(iv) $\dfrac{d}{dt}[\vec{r}(t) \cdot \vec{u}(t)] = \vec{r}'(t) \cdot \vec{u}(t) + \vec{r}(t) \cdot \vec{u}'(t)$ (dot product)
(v) $\dfrac{d}{dt}[\vec{r}(t) \times \vec{u}(t)] = \vec{r}'(t) \times \vec{u}(t) + \vec{r}(t) \times \vec{u}'(t)$ (cross product)
(vi) $\dfrac{d}{dt}\vec{r}(f(t)) = \vec{r}'(f(t))\, f'(t)$ (chain rule)
(vii) If $\vec{r}(t) \cdot \vec{r}(t) = c$, then $\vec{r}(t) \cdot \vec{r}'(t) = 0$.

Definition 1. Unit tangent vector at t:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

provided $\vec{r}'(t) \neq \vec{0}$.

Definition 2 (Integrating Vector-Valued Functions)**.** Let f, g, and h be real-valued functions on the closed interval $[a, b]$ and let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$.

(a) The *indefinite integral of* $\vec{r}$ is

$$\begin{aligned} \int \vec{r}(t)\, dt &= \int \langle f(t), g(t), h(t) \rangle\, dt \\ &= \left\langle \int f(t)\, dt, \int g(t)\, dt, \int h(t)\, dt \right\rangle . \end{aligned}$$

(b) The *definite integral of* $\vec{r}$ is

$$\begin{aligned} \int_a^b \vec{r}(t)\, dt &= \int_a^b \langle f(t), g(t), h(t) \rangle\, dt \\ &= \left\langle \int_a^b f(t)\, dt, \int_a^b g(t)\, dt, \int_a^b h(t)\, dt \right\rangle . \end{aligned}$$

The definition of the integral of a vector-valued function into 2-space is analogous.

Exercise 1. Suppose a particle is traveling along a curve represented by a vector-valued function. Show that if the speed of the particle is constant, then the velocity function is always perpendicular to the acceleration function.