Solution

Math 8 Spring 2018
Midterm I

Your name: ______________________

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INSTRUCTIONS

Except on problems that specify short answer, you must show your work in order to receive credit.

You have 2 hours.

This is a closed book, closed notes exam. Use of calculators is not permitted.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Good luck!

(1) *(12 points)* Calculate the fifth degree Taylor polynomial $T_5(x)$ of the function

$$f(x) = \left(\frac{x}{2} - 1\right)^6 \quad \text{centered at} \quad a = 4.$$

$f^{(k)}(x)$	$f^{(k)}(x)$	$f^{(k)}(4)$	$f^{(k)}(4)/k!$
0	$(x/2 - 1)^6$	1	1
1	$3(x/2 - 1)^5$	3	3
2	$\frac{15}{2}(\frac{x}{2} - 1)^4$	$15/2$	$15/4$
3	$15(x/2 - 1)^3$	15	$15/3! = \frac{5}{2}$
4	$45/2\,(x/2 - 1)^2$	$45/2$	$45/2 \cdot 4! = \frac{15}{2\cdot 4\cdot 2}$
5	$\frac{45}{2}(x/2 - 1)$	$45/2$	$\frac{45}{2}/5! = \frac{3}{2\cdot 4\cdot 2}$

$$T_5(x) = 1 + 3(x-4) + \frac{15}{4}(x-4)^2 + \frac{5}{2}(x-4)^3$$
$$+ \frac{15}{16}(x-4)^4 + \frac{3}{16}(x-4)^5$$

(2) *(12 points)* Determine whether the series is convergent or divergent. If it is convergent, find its sum. Justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{5 \cdot 2^{n-1}}{3^n}$.

(geometric series)

$$\sum_{n=1}^{\infty} \frac{5 \cdot 2^{n-1}}{3^n} = \frac{5}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

$$= \frac{5}{2}\left(\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - 1\right)$$

$$= \frac{5}{2}\left(\frac{1}{1-2/3} - 1\right) = \boxed{5}$$

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(b) $\sum_{n=1}^{\infty} \frac{n^2+n}{n^2-1}$.

$$\lim_{n\to\infty} \frac{n^2+n}{n^2-1} = \lim_{n\to\infty} \frac{1+1/n}{1-1/n^2} = 1 \ \boxed{\neq 0}$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ does not converge,

as $\lim_{n\to\infty} a_n \neq 0$

(3) *(12 points)* Find the radius of convergence of the following series. Show the steps of your solution and justify your answer.

$$\sum_{n=0}^{\infty} \frac{(n+1)\cdot x^{2n+1}}{4^n}.$$

Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)+1)}{n+1} \frac{x^{2(n+1)+1}}{x^{2n+1}} \frac{4^n}{4^{n+1}}$$

$$= \frac{n+2}{n+1} \frac{x^2}{4} \cdot 2$$

$$\lim_{n\to\infty} \left| \frac{n+2}{n+1} \frac{x^2}{4} \right| = \frac{x^2}{4} \underbrace{\lim_{n\to\infty} \frac{n+2}{n+1}}_{=1} = \frac{x^2}{4}$$

Convergence for $\frac{x^2}{4} < 1$ Div. for $\frac{x^2}{4} > 1$

$$\frac{x^2}{4} < 1 \iff x^2 < 4$$
$$\iff |x| < 2$$

Hence the radius of convergence is $\boxed{R = 2}$

(4) *(15 points)*

(a) By direct computation, find the third degree Taylor polynomial $T_3(x)$ of the function

$$\cos(x) \quad \text{centered at} \quad a = \frac{\pi}{4}.$$

k	$f^{(k)}(x)$	$f^{(k)}(\pi/4)$	$f^{(k)}(\frac{\pi}{4})/k!$
0	$\cos(x)$	$1/\sqrt{2}$	$1/\sqrt{2}$
1	$-\sin(x)$	$-1/\sqrt{2}$	$-1/\sqrt{2}$
2	$-\cos(x)$	$-1/\sqrt{2}$	$-1/\sqrt{2}\cdot 2$
3	$\sin(x)$	$1/\sqrt{2}$	$-1/\sqrt{2}\cdot 3!$
4	$\cos(x)$		

$$T_3(x) = \frac{1}{\sqrt{2}}\left(1 - (x-\pi/4) - \frac{1}{2}(x-\frac{\pi}{4})^2 + \frac{1}{6}(x-\pi/4)^3\right)$$

(b) Use Taylor's inequality to find an upper bound for the size $|R_3(x)|$ of the error in approximating $\cos(x)$ by the Taylor polynomial $T_3(x)$ above when $|x - \frac{\pi}{4}| \leq 0.2$.

1.) Upper bound for

$$|f^{(4)}(x)| = |\cos(x)| = M = 1 \quad \text{for } |x - \frac{\pi}{4}| \leq 0.2$$

$$2.)\ |R_3(x)| \leq \frac{M}{4!}|x - \frac{\pi}{4}|^4 \leq \frac{1}{4!}\cdot 0.2^4$$

(5) *(13 points)* Find the power series $\sum_{n=0}^{\infty} c_n \cdot (x-a)^n$ centered at the point a of the following functions.

(a) $f(x) = -x^2 \cdot \ln(1-x)$ for $a = 0$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

2

$$\int \frac{1}{1-x} = -\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

$$-\ln(1-0) = 0 = \sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1} + C = C$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x) \quad 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x^{n+3}}{n+1} = -x^2 \ln(1-x) \quad 2$$

(b) $g(x) = \dfrac{2}{1+3(x+1)^3}$ for $a = -1$.

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$$

$$\sum_{n=0}^{\infty} \left(-3(x+1)^3\right)^n = \frac{1}{1+3(x+1)^3}$$

$$\Rightarrow \sum_{n=0}^{\infty} 2 \cdot (-1)^n 3^n (x+1)^{3n} = \frac{2}{1+3(x+1)^3}$$

(6) *(12 points)* Evaluate the following three limits using an appropriate power series expansion.

(a) $\lim_{x\to 0} \frac{x^2 \cdot \cos(x^2) - x^2 + \frac{x^4}{2}}{x^{\mathbf{n}}}$ for $\mathbf{n} = 2$ and $\mathbf{n} = 4$.

$$\cos(y) = 1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots$$

$$y = x^2 \qquad x^2 \cos(x^2) = x^2\left(1 - \frac{x^4}{2} + \frac{x^8}{4!} \cdots\right)$$

$$x^2 \cos(x^2) - x^2 + \frac{x^4}{2} = \frac{x^4}{2} - \frac{x^6}{2} + \frac{x^8}{4!} \cdots$$

$$\lim_{x\to 0} \frac{x^2 \cdot \cos(x^2) - x^2 + \frac{x^4}{2}}{x^2} = \lim_{x\to 0} \frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{4!} + \cdots = 0$$

$$\lim_{x\to 0} \frac{x^2 \cos(x^2) - x^4 + \frac{x^4}{2}}{x^4} = \lim_{x\to 0} \frac{1}{2} - \frac{x^2}{2} + \frac{x^4}{4!} \cdots = \frac{1}{2}$$

(b) $\lim_{x\to 0} \frac{e^x + e^{-x}}{x}$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \cdots$$

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{x} = \lim_{x\to 0} \frac{2}{x} + x + \frac{2x^3}{4!} \cdots$$

The limit does not exist!

(7) *(12 points)* Let $\Delta = \Delta ABC$ be the triangle with vertices

$$A = (1,2,1)\,,\ B = (2,0,2) \ \text{ and } \ C = (3,1,5).$$

(a) Calculate the lengths of all three sides of the triangle Δ.

$\vec{AB} = B - A = (1,-2,1)$

$|\vec{AB}| = \sqrt{6}$.

$\vec{AC} = C - A = (2,-1,4)$

$|\vec{AC}| = \sqrt{21}$

$\vec{BC} = C - B = (1,1,3)$

$|\vec{BC}| = \sqrt{11}$

(b) Find the cosine of the angle $\alpha = \angle(\mathbf{u}, \mathbf{v})$ between the two vectors $\mathbf{u} = \langle 1,3,1 \rangle$ and $\mathbf{v} = \langle 2,-1,-3 \rangle$.

$\vec{u} \circ \vec{v} = 2 - 3 - 3 = -4$

$|\vec{v}| = \sqrt{14}$

$|\vec{u}| = \sqrt{11}$

$\cos(\alpha) = \dfrac{-4}{\sqrt{14} \cdot \sqrt{11}}$

(8) *(12 points)* **Short answer** You do not have to show your work.However, if you are not sure of your answer, you might want to explain your reasoning.

Fill in the blanks with the letter that corresponds to the description of the region in $\mathbb{R}^3$ represnted by given equation(s) or inequality.

Note: Some descriptions may be used more than once and others might not be used at all.

Region in $\mathbb{R}^3$

(a) E $1 < (x-1)^2 + (y-2)^2 + (z+1)^2 \leq 4$
(b) B $x = 2 - z,\ y = 5$
(c) H $x = 3z^2$
(d) A $x^2 + z^2 = 9$
(e) G $1 < z$
(f) B $y = 4,\ z = 1$

Description

(A) a cylinder
(B) a line
(C) a plane
(D) a sphere
(E) a ball with a smaller ball cut out around its center
(F) a point
(G) a half-space, meaning the space above or below a fixed plane
(H) a half-pipe or gutter