1) a) $f(x,y)=x^2+y^2$, $\quad xy=1$

$2x=\lambda y \qquad y=1/x$ so $2x=\frac{\lambda}{x} \Rightarrow x^2=\lambda/2 \qquad x^2=y^2 \qquad x^4=1 \qquad$ If $x=+1,\ y=+1$

$2y=\lambda x \qquad x=1/y$ so $2y=\frac{\lambda}{y} \Rightarrow y^2=\lambda/2 \qquad =\frac{1}{x^2}$ so $x=\pm 1, \qquad x=-1,\ y=-1$

$xy=1$

Have min $(1,1)$ and $(-1,-1)$. The max is unbounded, e.g. $x=\frac{1}{100},\ y=100$ (keep adding zeros)

b) $f(x,y)=y^2-x^2$ with $x^2+4y^3=4$

① $-2x=\lambda 2x \qquad x=0$ or $\lambda=-1$

② $2y=12\lambda y^2 \qquad y=0$ or $y=\frac{1}{6\lambda}$

③ $x^2+4y^3=4$

If $x=0$, $y=1$ by ③ $\qquad f(0,1)=1$ max

If $y=0$, $x=\pm 2$ by ③ $\qquad f(\pm 2,0)=-4$

If $\lambda=-1$, $y=\frac{-1}{6}$ so $x=\pm\sqrt{4+4/6^3}$ by ③

$f(\pm\sqrt{4+4/6^3},-\frac{1}{6})=\frac{6}{6^3}-4-\frac{4}{6^3}$
$=\frac{2}{6^3}-4$

No global min as f can go to $-\infty$ (region is unbounded)
e.g. $y=-100$, so $x=\pm\sqrt{4000004}$,
$f=10000-4000004$

c) $f(x,y)=e^{x+y}$, $\quad x^2+y^2=1$

$e^{x+y}=\lambda 2x \qquad$ so $x=y$

$e^{x+y}=\lambda 2y \qquad 2x^2=1$

$x^2+y^2=1 \qquad x=\pm\frac{\sqrt{2}}{2}$

Have
max $(\sqrt{2}/2,\sqrt{2}/2)$
min $(-\sqrt{2}/2,-\sqrt{2}/2)$

d) $f(x,y,z)=xy^2z \qquad x^2+y^2+z^2=1$

① $y^2z=\lambda 2x \qquad y^2 \overset{①}{=} \lambda 2\frac{x}{z} \overset{③}{=} \lambda 2\frac{z}{x}$

② $2xyz=\lambda 2y \qquad$ so $x^2=z^2$

③ $xy^2=\lambda 2z$

④ $x^2+y^2+z^2=1$

$y\neq 0$ as f can be >0, <0

so $xz \overset{②}{=} \lambda$,

$y^2=2xz\frac{x}{z}=2x^2$

so ④ $x^2+2x^2+x^2=1, \qquad x=\pm\frac{1}{2},\ z=\pm\frac{1}{2},\ y=\pm\frac{\sqrt{2}}{2}$

Max at $(\frac{1}{2},\pm\frac{\sqrt{2}}{2},\frac{1}{2}),\ (-\frac{1}{2},\pm\frac{\sqrt{2}}{2},-\frac{1}{2}) \qquad f=1/8$

Min at $(\frac{1}{2},\pm\sqrt{2}/2,-1/2),\ (-\frac{1}{2},\pm\sqrt{2}/2,1/2) \qquad f=-1/8$

e) $f(x,y,z)=2x+3y+5z$, $\quad x^2+y^2+z^2=38$

① $2=\lambda 2x \qquad x=2/2\lambda$

② $3=\lambda 2y \qquad y=3/2\lambda$

③ $5=\lambda 2z \qquad z=5/2\lambda$

④ $x^2+y^2+z^2=38$

④ $\frac{4}{4\lambda^2}+\frac{9}{4\lambda^2}+\frac{25}{4\lambda^2}=38$

$\frac{38}{4\lambda^2}=38 \Rightarrow \lambda^2=\frac{1}{4},\ \lambda=\pm 1/2$

$\lambda=1/2$ get max $f(2,3,5)=38$

$\lambda=-1/2$ get min $f(-2,-3,-5)=-38$

(note, these parallel to normal)

2) Let x=radius, y=height. Want min $f(x,y)=\underbrace{2\pi x^2}_{\text{base, top}}+\underbrace{2\pi xy}_{\text{sides}}$ with $\pi x^2 y=8$

① $4\pi x+2\pi y=\lambda 2\pi xy \Rightarrow 2x+y=\lambda xy \Rightarrow 2x+y=2y$ so $2x=y$

② $2\pi x=\lambda\pi x^2 \qquad x\neq 0$ so $x=2/\lambda$

③ $\pi x^2 y=8$

by ③, have radius height
$2\pi x^3=8 \Rightarrow x=\sqrt[3]{4/\pi},\ y=\frac{1}{2}\sqrt[3]{4/\pi}$

3) Will consider distance squared $f(x,y) = (x-2)^2 + (y-1)^2$ with $x^2+y^2=5$

① $2x-4 = \lambda 2x$ so $(1-\lambda)x = 2$, $x = \frac{2}{1-\lambda}$ by ③ $\frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda^2)} = 5$

② $2y-2 = \lambda 2y$ $(1-\lambda)y = 1$ so $y = \frac{1}{1-\lambda}$

③ $x^2+y^2=5$ $\Rightarrow (1-\lambda)^2 = 1$ so $\lambda = 0$ or $\lambda = 2$

If $\lambda=0$, have $(2,1)$. $f(2,1)=0$ as it is on the circle (could see this with no calculus)

If $\lambda=2$, have $(-2,-1)$, $f(-2,-1) = 17$. This is the antipodal point on the circle. (could see this with no calc)

4) Will consider distance squared $f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$ with $x^2+y^2-z^2=0$

① $2x-8 = \lambda 2x$

② $2y-4 = \lambda 2y$

③ $2z = -\lambda 2z$ so $z=0$ or $\lambda=-1$

④ $x^2+y^2-z^2=0$ if $z=0$, have $x=0, y=0$

If $\lambda=-1$,

by ① $x=2$, by ② $y=1$, by ④ $z^2=5$ so $z=\pm\sqrt{5}$

get $(2,1,\sqrt{5})$ and $(2,1,-\sqrt{5})$

as in HW #9

5) Maximize Volume $f(x,y,z) = xyz$ with $x^2+y^2+z^2=9$.

① $yz = \lambda 2x$ $x \cdot$① $xyz = \lambda 2x^2$ so $x^2=y^2=z^2$. Since sides are >0,

② $xz = \lambda 2y$ $y \cdot$② $xyz = \lambda 2y^2$ see $x=y=z$ (a cube!)

③ $xy = \lambda 2z$ $y \cdot$③ $xyz = \lambda 2z^2$ By ④, $3x^2=9$, $x=\sqrt{3}$

④ $x^2+y^2+z^2=9$ max volume $= (\sqrt{3})^3 = 3\sqrt{3}$