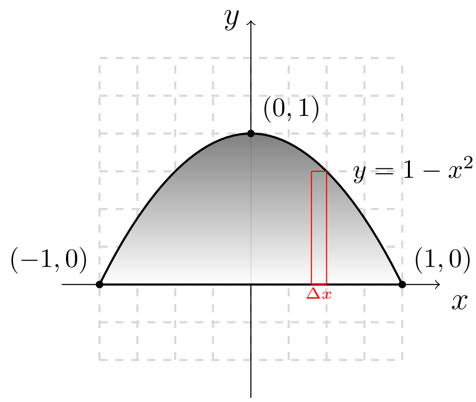
Math 8 — Sample Homework Solution

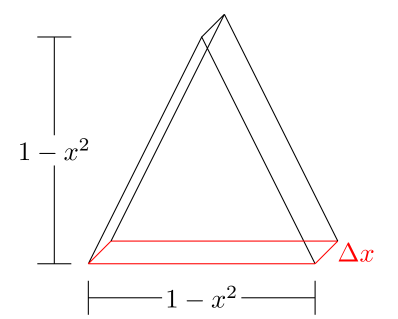
The base of a solid, S, is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis. The cross sections of S which are perpendicular to the x-axis are isosceles triangles with the base equal to the height. Find the volume of S.

Solution:

We first want to draw a picture of the base of S:



The shaded region is the base of S, and the red rectangle represents the base of an arbitrary cross section. We'll find the volume of this arbitrary cross section:



We know the cross section is an isosceles triangle where the base is equal to the height. (We actually do not need the information that the triangle is isosceles, since its area only depends on the base and height.) The area of a triangle is one half the product of the base and the height, and to find the volume of this slice we multiply that area by the thickness, Δx. So,

$$\begin{aligned} V_{\text{slice}} &= \frac{1}{2} bh\Delta x \\ &= \frac{1}{2}(1-x^2)(1-x^2)\Delta x \\ &= \frac{1}{2}(1-x^2)^2\Delta x. \end{aligned}$$

We could find an approximate volume, if we "filled" the solid with these slices and added up their volumes, which would look something like this:

$$V \approx \sum_i \frac{1}{2}(1 - x_i^2)^2 \Delta x.$$

When we take the limit, as the length Δx approaches zero, we have the exact volume in the form of the integral given below.

$$V = \int_{-1}^{1} \frac{1}{2}(1 - x^2)^2 \, dx.$$

Now, we need only to calculate the value of the integral:

$$\begin{aligned}
V &= \int_{-1}^{1} \frac{1}{2}(1 - x^2)^2 \, dx \\
&= \frac{1}{2} \int_{-1}^{1} \left(1 - 2x^2 + x^4\right) dx \\
&= \frac{1}{2} \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} \\
&= \frac{1}{2} \left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right] \\
&= \frac{1}{2} \left(\frac{8}{15} - \left(-\frac{8}{15}\right)\right) \\
&= \frac{8}{15}
\end{aligned}$$

Thus, the volume of the solid S is $\frac{8}{15}$ cubic units.