1. (12) Determine whether the following integral is convergent or divergent:

$$\int_0^{\infty} x^3 e^{-x^4}\, dx.$$

$$\int_0^{\infty} x^3 e^{-x^4} dx = \lim_{t \to \infty} \int_0^{t} x^3 e^{-x^4} dx$$

$$\int_0^{t} x^3 e^{-x^4} dx = \frac{1}{4} \int_0^{t^4} e^{-u} du$$

$u = x^4$

$du = 4x^3 dx$

$$= \frac{1}{4} \left. \frac{e^{-u}}{-1} \right]_0^{t^4}$$

$$= -\frac{1}{4} \left[e^{-t^4} - 1 \right]$$

$$\lim_{t \to \infty} -\frac{1}{4} \left(e^{-t} - 1 \right) = \frac{1}{4}$$

Hence $\int_0^{\infty} x^3 e^{-x^4} dx$ is cgt.

2. (10) Evaluate

$$\int_1^2 \frac{\ln x}{x^2}\, dx.$$

$u = \ln x \qquad dv = x^{-2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$

$$\int_1^2 \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x\Big]_1^2 - \int_1^2 -\frac{1}{x^2}\,dx$$

$$= -\left[\tfrac{1}{2}\ln 2 - \underbrace{\ln 1}_{=0}\right] - \left[\tfrac{1}{2} - 1\right]$$

$$= -\tfrac{1}{2}\ln 2 + \tfrac{1}{2}$$

$$= \tfrac{1}{2}(1 - \ln 2)$$

3. (14) Evaluate

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$$

$x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta$

$$\int \frac{1}{\sec^3\theta \sqrt{\sec^2\theta - 1}} \sec\theta \tan\theta \, d\theta$$

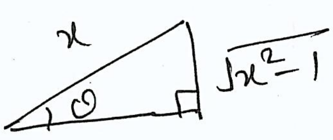
$$= \int \frac{\tan\theta}{\sec^2\theta \tan\theta} d\theta = \int \cos^2\theta \, d\theta$$

$$= \int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \, \cancel{d} + C$$

$$= \frac{1}{2}\theta + \frac{2\sin\theta\cos\theta}{4} + C$$

$$= \frac{1}{2}\theta + \frac{\sin\theta\cos\theta}{2} + C$$



$$= \frac{1}{2}\sec^{-1}x + \frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x} \cdot \frac{1}{x}\right) + C$$

$$= \frac{1}{2}\sec^{-1}x + \frac{1}{2}\frac{\sqrt{x^2-1}}{x^2} + C$$

4. (12) Evaluate

$$\int \sin^3 x \cos^2 x \, dx$$

$$= \int \sin^2 x \sin x \cos^2 x \, dx$$

$$= \int (1-\cos^2 x) \cos^2 x \sin x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1-u^2) u^2 \, du = -\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + c$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

5. (12) Determine if the following series $\sum_{n=1}^{\infty} \frac{(-1)^n n^3 2^n}{n!}$ converges. Mention any test(s) that you might use and verify that it is applicable.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^3 2^{n+1}}{(n+1)!} \frac{n!}{n^3 2^n} \right|$$

$$= \lim_{n\to\infty} \frac{2}{n+1} \left(\frac{n+1}{n}\right)^3$$

$$= \lim_{n\to\infty} 2 \frac{(n+1)^2}{n^3} = 0 < 1$$

Hence by Ratio test, the series is absolutely convergent & hence it cgt.

6. (12) Determine whether the series is covergent or divergent. Mention any test(s) that you might use and verify that it is applicable.

$$\sum_{n=1}^{\infty} \frac{1 + \sin n}{5^n}$$

$\sum_{n=1}^{\infty} \frac{1}{5^{-n}}$ is a geom. series with

$r = 1/5 < 1$

& hence cgt.

$$\left| \frac{\sin n}{5^n} \right| \leq \frac{1}{5^{-n}}$$

hence $\sum \frac{\sin n}{5^n}$ is absolute convergent

by comparison test & $\sum \frac{1}{5^n}$ is cgt.

Hence $\sum_{n=1}^{\infty} \frac{1 + \sin n}{5^{-n}}$ is cgt, [illegible]

$$\left(\& \ \sum \frac{1 + \sin n}{5^n} = \sum \frac{1}{5^n} + \sum \frac{\sin n}{5^n} \right)$$

7. (8) Determine if the following series $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+n}}{n-1}$ converges. Mention any test(s) that you might use and verify that it is applicable.

$$\lim_{n\to\infty} \frac{\sqrt{n^2+n}}{n-1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+1/n}}{1-1/n}$$

$$= 1 \neq 0$$

By test for divergence, Σ is dgt

8. (20) For each of the following statements, fill in the blank with the letters **T** or **F** depending on whether the statement is true or false. You do not need to show your work and no partial credit will be given on this problem.

(a) Let $\{a_n\}$ and $\{b_n\}$ be two sequences with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 2$ and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} b_n$ is convergent.

limit comparison test

ANS: T

(b) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent.

$\sum \frac{1}{\sqrt{n}}$ is not cgt

but $\sum \frac{(-1)^n}{\sqrt{n}}$ is cgt

by AST

ANS: T

(c) The sequence $\{\sin(\pi/n)\}$ is divergent.

$\lim_{n\to\infty} \sin(\pi/n)$

$= \sin\left(\lim \pi/n\right)$

$= \sin 0 = 0$

ANS: F

(d) If f is a positive, continuous and decreasing function on $[1, \infty)$, then

$$\sum_{n=1}^{\infty} f(n) = \int_{1}^{\infty} f(x)\, dx$$

ANS: F

(e) The sequence $\{\ln(n+1) - \ln n\}$ converges to 1.

$$\lim_{n\to\infty} \ln(n+1) - \ln n$$

$$= \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)$$

$$= 0$$

ANS: F