

1. A tank contains 1000 liters of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 liters/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank

(a) after t minutes?

(b) after 20 minutes?

Answer. Let us define the function

$$s(t) = \text{kgs of salt in the tank at time } t \text{ minutes.}$$

This is the function we need to find. Now, obviously, we don't know what $s(t)$ is, but we *do* know its derivative:

$$\begin{aligned} s'(t) &= \text{the rate at which the amount of salt in the tank is changing} \\ &= (\text{rate of salt going in}) - (\text{rate of salt going out}) \end{aligned}$$

First, how much salt is going into the tank per minute?

$$\frac{0 \text{ kgs of salt}}{\text{liter}} \times \frac{10 \text{ liters}}{\text{min}} = \frac{0 \text{ kgs of salt}}{\text{min}}$$

Why the 0 kgs? Because pure water (i.e. no salt in it) is entering the tank.

Now how much salt is leaving the tank? Well, how many kgs of salt are there per liter in the tank (which, remember, is kept thoroughly mixed)? This many: $\frac{s(t) \text{ kgs}}{1000 \text{ liters}}$. And how fast is the solution being drained from the tank? Solution is leaving the tank at the rate of $\frac{10 \text{ liters}}{\text{min}}$. Therefore, the rate at which salt is leaving the tank is

$$\frac{s(t) \text{ kgs of salt}}{1000 \text{ liters}} \times \frac{10 \text{ liters}}{\text{min}} = \frac{\frac{1}{100}s(t) \text{ kgs of salt}}{\text{min}}.$$

Therefore, the differential equation describing this situation is

$$\begin{aligned} s'(t) &= (\text{rate of salt going in}) - (\text{rate of salt going out}) \\ &= 0 - \frac{1}{100}s(t) \\ &= -\frac{1}{100}s(t) \end{aligned} \tag{1}$$

Solving equation (1) for $s(t)$:

$$\frac{ds}{dt} = -\frac{1}{100}s \implies \frac{ds}{s} = -\frac{1}{100}dt \implies s(t) = Ce^{-1/100t}$$

Here, C is an arbitrary constant that we need to determine. The problem states that, at the start, there are 15 kgs of dissolved salt in the tank. Therefore, our initial condition is $s(0) = 15$. That being the case,

$$s(0) = 15 = Ce^{-1/100 \cdot 0} \implies 15 = C.$$

Therefore, the answer to part (a) is $s(t) = 15e^{-1/100t}$. The answer to part (b) is

$$s(20) = 15e^{-1/100 \cdot 20} = 15e^{-1/5} \approx 12.28 \text{ kgs.}$$

2. A tank contains 1000 liters of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 liters/min. Brine that contains 0.04 kg of salt per liter of water enters the tank at the rate of 10 liters/min. The solution is kept thoroughly mixed and drains from the tank at 15 liters/min. How much salt is left in the tank

- (a) after t minutes?
 (b) after 1 hour?

Answer. The set up is going to be similar to the previous problem. We first define the function

$$s(t) = \text{kgs of salt in the tank at time } t \text{ minutes.}$$

and we know that

$$\begin{aligned} s'(t) &= \text{the rate at which the amount of salt in the tank is changing} \\ &= (\text{rate of salt going in}) - (\text{rate of salt going out}). \end{aligned}$$

The things we have to determine are the “rate in” and “rate out” quantities. Unlike the first problem, in this one there are *two* sources of brine solution entering the tank, one at the rate of 5 liters/min and the other at the rate of 10 liters/min. So “rate in” will be the sum of two quantities:

$$\begin{aligned} \text{rate in} &= \left(\frac{0.05 \text{ kgs of salt}}{\text{liter}} \times \frac{5 \text{ liters}}{\text{min}} \right) + \left(\frac{0.04 \text{ kgs of salt}}{\text{liter}} \times \frac{10 \text{ liters}}{\text{min}} \right) \\ &= \left(\frac{0.25 \text{ kgs of salt}}{\text{min}} \right) + \left(\frac{0.4 \text{ kgs of salt}}{\text{min}} \right) \\ &= \frac{0.65 \text{ kgs of salt}}{\text{min}} \end{aligned}$$

And with the same reasoning we used in the first problem, the “rate out” is

$$\frac{s(t) \text{ kgs of salt}}{1000 \text{ liters}} \times \frac{15 \text{ liters}}{\text{min}} = \frac{3}{200} s(t) \frac{\text{kgs of salt}}{\text{min}}.$$

Therefore, the differential equation describing this situation is

$$\begin{aligned} s'(t) &= (\text{rate of salt going in}) - (\text{rate of salt going out}) \\ &= 0.65 - 0.015s(t) \end{aligned} \tag{2}$$

Rewriting equation (2), we have: $s'(t) + 0.015s(t) = 0.65$.

We’ve seen equations of this form in class: $y' + ay = g(x)$, and we know the solution of such an equation: $y = e^{-ax} \int e^{ax} g(x) dx + De^{-ax}$.

Therefore, for this problem,

$$\begin{aligned} s(t) &= e^{-0.015t} \cdot (43.333 e^{0.015t}) + De^{-0.015t} \\ &= 43.333 + De^{-0.015t}, \end{aligned}$$

and using the initial condition $s(0) = 0$ (recall that the tank is filled with pure water at the start), we have $D = -43.333$.

Therefore, the answer to part (a) is $43.333 - 43.333 e^{-0.015t}$. The answer to part (b) is

$$s(60) = 43.333 - 43.333 e^{-0.015 \cdot 60} \approx 25.72 \text{ kgs.}$$

3. Into a 2000 liter container is placed 1000 liters of a brine solution containing 40 kg of salt. A brine solution containing .02 kg/l of salt flows into the container at a rate of 50 l/min. The solution is kept thoroughly mixed, and the mixture flows out at a rate of 25 l/min. How much salt is in the container at the moment it overflows?

Answer. The set up is going to be similar to the previous problems. We first define the function

$$s(t) = \text{kgs of salt in the tank at time } t \text{ minutes.}$$

and we know that

$$s'(t) = (\text{rate of salt going in}) - (\text{rate of salt going out}).$$

First, the “rate in”:

$$\text{rate in} = \frac{0.02 \text{ kgs of salt}}{\text{liter}} \times \frac{50 \text{ liters}}{\text{min}} = \frac{1 \text{ kg of salt}}{\text{min}}$$

Now for the “rate out,” and here we have to be a little bit careful. Note that more solution is flowing in than is flowing out of the tank. At the very beginning, how much solution is in the tank? 1000 liters. Ok, how much solution is in the tank after 1 minute? $1000 + 50$ (liters flowing in per minute) - 25 (liters flowing out per minute) = $1000 + (50 - 25) = 1025$ liters. And after 2 minutes? $1000 + (50 - 25) \cdot 2 = 1050$ liters. Therefore, after t minutes, there is $1000 + (50 - 25)t = 1000 + 25t$ liters of solution in the tank. Therefore, the “rate out” is

$$\frac{s(t) \text{ kgs of salt}}{(1000 + 25t) \text{ liters}} \times \frac{25 \text{ liters}}{\text{min}} = \frac{s(t) \text{ kgs of salt}}{(40 + t) \text{ min}}.$$

This leads to the differential equation:

$$s'(t) = 1 - \frac{s(t)}{40 + t} \implies s'(t) + \frac{s(t)}{40 + t} = 1.$$

This equation is of the general form $s'(t) + p(t)s(t) = g(t)$, and we know how to solve equations of this form:

$$s(t) = \frac{1}{\mu(t)} \left[\int \mu(t)g(t)dt \right] + D \frac{1}{\mu(t)} \quad \text{where} \quad \mu(t) = \exp \left(\int p(t)dt \right).$$

Plugging things in and integrating yields:

$$s(t) = \frac{80t + t^2}{2(40 + t)} + \frac{D}{40 + t}$$

At time $t = 0$, there is 40 kgs of salt in the tank, so $s(0) = 40$. This implies that $D = 1600$. Therefore,

$$s(t) = \frac{80t + t^2}{2(40 + t)} + \frac{40}{40 + t}$$

When (i.e. for what time t) does the tank overflow? The moment there is 2000 liters of solution in it. We have an expression for the amount of solution in the tank: $1000 + 25t$. So setting this equal to 2000 and solving for t : $1000 + 25t = 2000 \implies t = 40$.

Therefore, at the moment of overflow, there is $s(40) = 50$ kgs of salt in the container.