1 Matrix Multiplication, Part II

1.1 The Definition

The **product** of two matrices A and B is defined whenever A is an $m \times n$ matrix and B is an $n \times p$ matrix. That is, the number of columns of A has to be the same as the number of rows of B. This means that rows of A and columns of B have the same number of entries, so the following definition makes sense:

AB is an $m \times p$ matrix. Column j of AB is the product of A and column j of B. In other words, to find AB, we multiply A times all the columns of B, and those are the columns of the product. Putting this still another way, the entry in row i, column j of AB is the product of row i of A with column j of B.

Here are some examples:

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 12 \\ 3 & 5 & 34 \end{pmatrix}.$$

This is the product of a 2×2 matrix with a 2×3 matrix, so the result is a 2×3 matrix. The entry in row 1, column 3 is the product of the first row of the left-hand factor and the third column of the right-hand factor, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} (8, 2) = 12$. The third column of the product is the left-hand factor times the third column of the right-hand factor:

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 34 \end{pmatrix}.$$

The product

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

is not defined. This is the product of a 2×3 matrix with a 2×2 matrix; the number of columns of the first factor (3) is not the same as the number of rows of the second factor (2), so we cannot carry out the multiplication. You can see that if you try to take the product of a row of the left-hand matrix with a column of the right-hand matrix, the numbers of entries don't match.

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -5 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 1 & 0 \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

1.2 Rules for Matrix Arithmetic

The examples in the preceding section should make clear that matrix multiplication is not completely like multiplication of numbers. In particular, it is important to remember that matrix multiplication is in general *not commutative*. Even if A and B are both square ($n \times n$) matrices, so the products AB and BA are both defined and have the same dimensions ($n \times n$), it will usually be the case that $AB \neq BA$.

Another way in which matrix multiplication differs from multiplication of numbers, which we have already seen, is the following: It is possible for some non-zero matrices A to have $AB = AC$ but $B \neq C$, or $BA = CA$ but $B \neq C$. In other words, we can't "cancel out" A in the equation $AB = AC$.

Also, it is possible for neither A nor B to be zero, but for AB to be zero.

There's some good news as well. Many of the familiar laws for arithmetic of numbers do hold in the case of matrix arithmetic. In order to say what these are, we will first define two other operations on matrices, which we have not used so far but which will be useful in many contexts.

The **sum** of two matrices A and B is defined when A and B have the same dimensions. To add A and B, add their corresponding entries:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+1 \\ 3-1 & 4-2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}.$$

The **product** of a scalar (number) c and a matrix A is computed by multiplying each entry of A by c:

$$3\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3(1) & 3(2) \\ 3(-1) & 3(0) \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix}.$$

These operations on matrices are just like the corresponding operations on vectors and obey the same sorts of rules:

$$A + B = B + A$$
$$A + (B + C) = A + (B + C)$$
$$c(A + B) = (cA) + (cB)$$
$$c(dA) = (cd)A$$

Other rules apply to their interaction with matrix multiplication:

$$A(B + C) = (AB) + (AC)$$
$$(B + C)A = (BA) + (CA)$$
$$c(AB) = (cA)B = A(cB)$$

Finally, an important property of matrix multiplication: Matrix multiplication is associative:

$$A(BC) = (AB)C$$

In fact, the definition of matrix multiplication is designed to make sure that matrix multiplication is associative. This is because the mathematicians who came up with this definition were interested in linear functions of the sort

$$L_A(X) = AX.$$

Suppose we have two such linear functions, given by two matrices A and B. Let's look now at the composition,

$$(L_A \circ L_B)(X) = L_A\big(L_B(X)\big) = L_A(BX) = A(BX).$$

Given that matrix multiplication is associative, we can continue with

$$(L_A \circ L_B)(X) = A(BX) = (AB)X = L_{AB}(X).$$

In other words, the composition of linear functions is another linear function, and its matrix is the product of the matrices of the original functions.

If you study multivariable calculus, you will see that this gives a nice "chain rule" for derivatives of functions from n-dimensional space to m-dimensional space.

2 Matrix Inverses

In an earlier handout, we hinted about the possibility of solving a matrix equation $AX = B$ by multiplying both sides of the equation by an inverse A^{-1} to the matrix A. Now we are going to define the inverse matrix and see how to compute it. First, we need to define another concept:

The $n \times n$ **identity matrix** I (sometimes denoted I_n, if the dimension is not clear from context) is the $n \times n$ matrix that has 1's down its main diagonal and 0's everyplace else:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \cdots$$

The identity matrices are the 1's (the "units" in technical mathematical language) of matrix multiplication. That is, if I is the $n \times n$ identity matrix, and A and B are any matrices of the right dimensions so the following products are defined, then

$$IA = A \qquad \text{and} \qquad BI = B.$$

Now an $n \times n$ matrix A has an **inverse** A^{-1} if there is an $n \times n$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

This means, in particular, that multiplying by A^{-1} is like dividing by A:

$$A^{-1}(AX) = (A^{-1}A)(X) = IX = X.$$

Now that we have a definition of a matrix inverse, how do we find one? First, we need a small but important fact:

Fact: If A and B are $n \times n$ matrices and $AB = I$, then also $BA = I$. (Remember that in general, $AB \neq BA$, so the truth of this fact is not obvious.)

This fact means that if we can find an $n \times n$ matrix X with the property that $AX = I$, then we will know that $XA = I$ also, so $X = A^{-1}$. And (guess what?) we know how to solve the matrix equation $AX = I$: Write down the augmented matrix $A \vdots I$, row-reduce it, then look at the equivalent matrix equation obtained from the new, row-reduced augmented matrix.

For example, let's try to find an inverse to the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

We try to solve the matrix equation $AX = I$ by writing down the augmented matrix

$$A \vdots I = \begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 1 & 1 & 1 & \vdots & 0 & 1 & 0 \\ 2 & 1 & 1 & \vdots & 0 & 0 & 1 \end{pmatrix}$$

and row-reducing it. (Some of these steps combine two operations in one. For example, the first step consists of first adding -1 times row 1 to row 2 and then adding -2 times row 1 to row 3. The second step is simply to multply row 2 by -1.)

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 1 & 1 & 1 & \vdots & 0 & 1 & 0 \\ 2 & 1 & 1 & \vdots & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & -1 & 0 & \vdots & -1 & 1 & 0 \\ 0 & -3 & -1 & \vdots & -2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 1 & -1 & 0 \\ 0 & -3 & -1 & \vdots & -2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & -1 & 2 & 0 \\ 0 & 1 & 0 & \vdots & 1 & -1 & 0 \\ 0 & 0 & -1 & \vdots & 1 & -3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & -1 & 2 & 0 \\ 0 & 1 & 0 & \vdots & 1 & -1 & 0 \\ 0 & 0 & 1 & \vdots & -1 & 3 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -1 \end{array}\right).$$

This is the augmented matrix of the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix},$$

or

$$X = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix}.$$

Therefore we have our solution:

$$A^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix}.$$

This is one way things can work out when we try to find the inverse of a matrix A. Here's another: let us try to find an inverse to the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix}.$$

We try to solve the matrix equation $AX = I$ by writing down the augmented matrix

$$A \vdots I = \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$$

and row-reducing it:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & -1 & 0 & \vdots & -1 & 1 & 0 \\ 0 & -1 & 0 & \vdots & -2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 1 & -1 & 0 \\ 0 & -1 & 0 & \vdots & -2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & -1 & 2 & 0 \\ 0 & 1 & 0 & \vdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \vdots & -1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & -1 & 2 & 0 \\ 0 & 1 & 0 & \vdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \vdots & 1 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & \vdots & 0 & 3 & -1 \\ 0 & 1 & 0 & \vdots & 0 & -2 & 1 \\ 0 & 0 & 0 & \vdots & 1 & 1 & -1 \end{pmatrix}.$$

This is the augmented matrix of the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 & 3 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Now this equation has no solutions. How do we know this? The entries in the third row of the product

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} X$$

will be the products of the third row of the first factor with the columns of X. Since the third row of the first factor consists entirely of zeroes, these products will all be zero as well, so whatever X is, the third row of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} X$$

will be $(0, 0, 0)$; it is impossible for it to be $(1, 1, -1)$. Therefore, the matrix A does not have an inverse.

These are the two possibilities. We can collect this information into a procedure:

> To find the inverse A^{-1} of a square matrix A, write down the matrix $A \vdots I$ and then row-reduce it. Either it will row-reduce to a matrix of the form $I \vdots B$, in which case $B = A^{-1}$, or it will row-reduce to a matrix of the form $C \vdots B$ where C has a row consisting entirely of zeroes, in which case A has no inverse.

2.1 Using Matrix Inverses

In order to know that we can solve a matrix equation $AX = B$ by multiplying through by A^{-1} (assuming that A has an inverse), we need to know the following fact:

Fact: If A is any matrix that has an inverse, then multiplying through any matrix equation by A gives an equivalent matrix equation.

We can see why this must be true. Suppose we have a matrix equation $BX = C$. If X is any solution to this, so in fact $BX = C$, X will still be a solution to the matrix equation $ABX = AC$, because multiplying the same matrix by A (whether it is called BX or called C) will give the same answer. So when we multiply a matrix equation by any matrix A, we don't lose any solutions.

On the other hand, in general, we might gain solutions. In an earlier handout, we saw that the matrix equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

has a unique solution,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

but the equation

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

or, multiplying out both sides,

$$\begin{pmatrix} x + 2y \\ 2x + 4y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix},$$

has as solutions all points on the line $x + 2y = 5$; so the second equation is not equivalent to the original one.

If A has an inverse (A is *invertible*), on the other hand, going from $BX = C$ to $ABX = AC$ cannot gain any solutions. Here's why: Suppose X is a solution to the new equation $ABX = AC$. Then X is also a solution to $A^{-1}ABX = A^{-1}AC$ (because multiplying through by any matrix, in particular A^{-1}, does not lose any solutions.) But since $A^{-1}A = I$, $A^{-1}ABX = A^{-1}AC$ is the same as $BX = C$. This means X was a solution to $BX = C$, the original equation.

Now we know that multiplying through a matrix equation by an invertible matrix gives an equivalent equation. This means that we can safely multiply $AX = B$ through by A^{-1} (since A^{-1} has an inverse, namely A) to get the equivalent equation $A^{-1}AX = A^{-1}B$, or $X = A^{-1}B$.

For example, consider the matrix equation

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}.$$

The matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

has an inverse, which we have already calculated, namely

$$A^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix}.$$

Therefore, we can solve the original matrix equation by multiplying through by this matrix to get

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix},$$

which, using the fact that $A^{-1}AX = IX = X$ to evaluate the left hand side and multiplying out the right hand side, we can see is just

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 10 \end{pmatrix}.$$

We can see that, if A is a square matrix and A has an inverse, then any matrix equation $AX = B$ will have exactly one solution, namely $X = A^{-1}B$.

On the other hand, suppose A is a square matrix with no inverse. This means that when we try to find A^{-1} by our row-reduction method, we cannot row-reduce A to the identity matrix, but instead row-reduce it to a matrix $\overline{A}$ that has a row entirely consisting of zeroes. This means that when we use row-reduction to solve the matrix equation $AX = B$, we end up with an equivalent equation $\overline{A}X = \overline{B}$ where $\overline{A}$ has a zero row. If $\overline{B}$ has a nonzero entry b someplace where $\overline{A}$ has a zero row, this means the associated system of linear equations includes the equation $0 = b$, so it has no solutions. Otherwise, the associated system of equations includes at least one equation $0 = 0$, and when we ignore that equation, there are more variables than equations, so there are infinitely many solutions.

This says that $n \times n$ matrices come in two types:

1. Matrices A with the properties:
   (a) A can be row-reduced to I.
   (b) A is invertible.
   (c) Any matrix equation $AX = B$ has a unique solution.
2. Matrices A with the properties:
   (a) A can be row-reduced to a matrix with a zero row.
   (b) A is not invertible.

(c) Any matrix equation $AX = B$ has either no solutions or infinitely many solutions.

This tells us, for example, that if we know some matrix equation $AX = B$ has no solutions, we also know immediately that A is not invertible, and that A cannot be row-reduced to I. In Section 3 we will add one more pair of distinguishing properties to this list.

3 Determinants.

We have already seen how to compute determinants of 3×3 matrices, which we use to compute cross products. Now we will see how to compute the determinant of an $n \times n$ matrix. The determinant of a square ($n \times n$) matrix is a number associated to that matrix. The determinant has several nice properties, but for the moment we will be especially interested in one:

The determinant of a square matrix is non-zero if and only if that matrix has an inverse. That means we can add one more property to our list of the properties of the two types of $n \times n$ matrices:

1. Matrices A with the properties:
   (a) A can be row-reduced to I.
   (b) A is invertible.
   (c) Any matrix equation $AX = B$ has a unique solution.
   (d) The determinant of A is not zero.
2. Matrices A with the properties:
   (a) A can be row-reduced to a matrix with a zero row.
   (b) A is not invertible.
   (c) Any matrix equation $AX = B$ has either no solutions or infinitely many solutions.
   (d) The determinant of A is zero.

We can see why this must be the case for 3×3 matrices by thinking geometrically. You may remember from the textbook that the scalar triple product of three vectors,

$$\vec{u} \cdot (\vec{v} \times \vec{w}),$$

is the determinant of the matrix A whose rows are the coordinates of $\vec{u}$, $\vec{v}$ and $\vec{w}$. In other words, the determinant of A is plus or minus (depending on whether $\vec{u}$, $\vec{v}$, $\vec{w}$ in that order have a right-handed or left-handed orientation) the volume of the parallelopiped with edges $\vec{u}$, $\vec{v}$ and $\vec{w}$. This tells us that the determinant of A is zero exactly in case $\vec{u}$, $\vec{v}$ and $\vec{w}$ are co-planar.

Now let's think about the matrix equation

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This corresponds to a system of linear equations. The first equation is the equation of the plane normal to $\vec{u}$ and containing the origin. (This is because if $\vec{u} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, then the first row of the matrix A is $\vec{u}$, or $(a\,b\,c)$, and so the first equation becomes $ax + by + cz = 0$.) The second equation is the equation of the plane normal to $\vec{v}$ and containing the origin, and the third equation is the equation of the plane normal to $\vec{w}$ and containing the origin.

If the determinant of A is zero, then the normal vectors to the three planes are coplanar, so there is a line through the origin normal to all three normal vectors. This line is the intersection of the three planes, so the matrix equation has infinitely many solutions. On the other hand, if the determinant of A is non-zero, the normal vectors to the three planes are not coplanar, so there is no line normal to all three normal vectors. Therefore there is no line lying in all three planes, so the planes intersect only in the single point $(0,0,0)$, and the matrix equation has a unique solution.

3.1 How to Compute Determinants:

Some of this is review for us. We're including it all here for completeness.

We define the determinant of a 2×2 matrix this way:

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Then we use our definition of the determinant of a 2×2 matrix to define the determinant of a 3×3 matrix:

$$det(A) = det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

In other words, we go across the first row of the matrix A, (a, b, c). We multiply each entry by the determinant of the 2×2 matrix we get from A by crossing out the row and column containing that entry. (Try this. If you take A and cross out the row and column containing b (the first row and the second column) you get the matrix $\begin{pmatrix} d & f \\ g & k \end{pmatrix}$; this was the matrix whose determinant we multiplied b by in computing the determinant of A.) Then we add and subtract the resulting terms, alternating signs (add the a-term, subtract the b-term, add the c-term.)

Example 1

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 4 \\ 8 & 4 & 10 \end{vmatrix} = (1)\begin{vmatrix} 0 & 4 \\ 4 & 10 \end{vmatrix} - (2)\begin{vmatrix} 3 & 4 \\ 8 & 10 \end{vmatrix} + (1)\begin{vmatrix} 3 & 0 \\ 8 & 4 \end{vmatrix} = 0.$$

We can use this same method to compute the determinant of a 4×4 matrix.

Example 2

$$\begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ -1 & -4 & -2 & -1 \\ 1 & 4 & 4 & 1 \end{vmatrix} =$$

$$(1)\begin{vmatrix} 2 & 0 & 0 \\ -4 & -2 & -1 \\ 4 & 4 & 1 \end{vmatrix} - (2)\begin{vmatrix} 0 & 0 & 0 \\ -1 & -2 & -1 \\ 1 & 4 & 1 \end{vmatrix} +$$

$$(2)\begin{vmatrix} 0 & 2 & 0 \\ -1 & -4 & -1 \\ 1 & 4 & 1 \end{vmatrix} - (1)\begin{vmatrix} 0 & 2 & 0 \\ -1 & -4 & -2 \\ 1 & 4 & 4 \end{vmatrix}$$

$$= 0.$$

The same method works for any square matrix of any size. And you don't have to use the first row; you can use any row or any column, as long as you know where to put the plus and minus signs. Here's how you know: Use this checkerboard pattern:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example: Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{pmatrix}.$$

Solution: The second column of this matrix has lots of zeroes, so that's a good one to use for computing the determinant. From our checkerboard matrix, we see that we should start with a minus sign going down the second column. So we compute the determinant of A as

$$-0\begin{vmatrix} 3 & 0 & 5 \\ 2 & 4 & -3 \\ 1 & 5 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & -3 \\ 1 & 5 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 5 \\ 1 & 5 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 5 \\ 2 & 4 & -3 \end{vmatrix} =$$

$$-\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 5 \\ 1 & 5 & 0 \end{vmatrix} = 30.$$

Finally, here's one more trick you can use for computing determinants: The elementary row operations change the determinant of a matrix in specific ways:

1. Multiplying a row of matrix by a number c multiplies its determinant by the same number.

2. Adding a multiple of one row of a matrix to another row *does not change* the determinant.

3. Interchanging two rows of a matrix changes the sign of the determinant.

Example: Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{pmatrix}.$$

Solution: First we add -1 times row 4 to row 1. This does not change the determinant, but it makes it easier to compute, because it gives us a matrix with more zero entries,

$$\begin{pmatrix} 0 & 0 & -3 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{pmatrix}.$$

Now we again expand along the second column. All terms but the third are zero, and the third term gives

$$-1 \begin{vmatrix} 0 & -3 & -1 \\ 3 & 0 & 5 \\ 1 & 5 & 0 \end{vmatrix} = 30.$$

Example: Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{pmatrix}.$$

We row-reduce the matrix to "upper triangular" form; that is, a form in which all the entries below the main diagonal are zero. First we add multiples of row 1 to the other rows to make all but the first entry in the first column equal to zero. This does not change the determinant.

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 0 & -6 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 1 \end{pmatrix}.$$

Now we interchange rows 2 and 3. *This changes the sign of the determinant* and we need to remember this.

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -6 & 8 \\ 0 & 0 & 3 & 1 \end{pmatrix}.$$

The second column now has all zeroes below the main diagonal, so we add a multiple of row 3 to row 4 to take care of the third column.

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -6 & 8 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

Now it is easy to compute the determinant of an upper triangular matrix; we just keep expanding along the first column, so all the terms but the first are zero:

$$\begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -6 & 8 \\ 0 & 0 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & -6 & 8 \\ 0 & 0 & 5 \end{vmatrix} = \begin{vmatrix} -6 & 8 \\ 0 & 5 \end{vmatrix} = -30.$$

But remember we interchanged two rows, which changes the sign of the determinant, so

$$det(A) = 30.$$

Exercise 1 *For each of the following matrix products, state whether the product is defined, and if it is, what the dimensions of the product will be. You do not have to do the multiplication.*

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & -1 & 0 \\ 2 & 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -1 & 0 \\ 2 & 5 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 5 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 5 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 5 & 2 \\ 4 & -4 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 & 5 \\ 3 & 6 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Exercise 2 *Carry out the following matrix multiplications.*

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 8 & 9 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 9 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 5 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 9 & 8 \end{pmatrix} \begin{pmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{pmatrix}$$

Exercise 3 *Find inverses for all of the following matrices that are invertible.*

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 4 & 9 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 4 & 7 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 3 & -3 \\ 0 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 5 & 2 & 7 \\ 3 & 5 & 2 & 7 \\ 11 & 4 & 3 & 15 \\ -4 & 5 & 19 & 3 \end{pmatrix}$$

Exercise 4 *Solve each of these systems of simultaneous linear equations. (Hint: Convert to a matrix equation $AX = B$, and check for the matrix A in the preceding exercise.)*

$$x + 2y + z = 3$$
$$2x + 3y + 3z = 3$$
$$4x + 9y + 6z = 3$$

$$x + 2y + z = 1$$
$$2x + 3y + 3z = 3$$
$$4x + 9y + 6z = 9$$

$$x + 2y + z = 6$$
$$2x + 3y + 3z = 12$$
$$4x + 9y + 6z = -3$$

$$x + 2y + z = 0$$
$$2x + 3y + 3z = 0$$
$$4x + 7y + 5z = 0$$

Exercise 5 *Does the system of simultaneous linear equations*

$$x + 2y + z = 0$$

$$3x + 4z = 0$$

$$8x + 4y + 10z = 0$$

have no solutions, one solution, or infinitely many solutions? How do you know?

Exercise 6 *Put the matrix*

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ -1 & -4 & -2 & -1 \\ 1 & 4 & 4 & 1 \end{pmatrix}$$

into row echelon form. Then say whether or not the determinant of this matrix is zero.

Exercise 7 *Find the determinant of the matrix*

$$\begin{pmatrix} 2 & 1 & 4 & 8 \\ 0 & 2 & 5 & 19 \\ 0 & 0 & 3 & -1 \\ 2 & 1 & 4 & 0 \end{pmatrix}.$$