1 Matrix Multiplication Part I

In this section we will introduce a new way of writing a system of linear equations as a single equation involving matrices. To do this we will define a special case of *matrix multiplication*. Although our motivation for introducing matrix multiplication now is to simplify our methods of dealing with systems of linear equations, matrix multiplication is actually an important mathematical tool that has many applications.

1.1 The Definition

Remember that a matrix consisting of only one column is sometimes called a column vector. Similarly, a matrix consisting of only one row is sometimes called a row vector. We define the **product** of a row vector with a column vector (*always* written in that order) the same way we defined the dot product of two vectors: Multiply together corresponding entries, and add the products:

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz.$$

We use this to define **multiplication** of a matrix by a column vector:

If A is an $n \times m$ matrix (with n rows and m columns) and X is an $m \times 1$ matrix (an m-dimensional column vector), then AX is the $n \times 1$ matrix (n-dimensional column vector) whose i^{th} entry is the product of the i^{th} row of A with X. For example:

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y - z \\ x + 3z \end{pmatrix}.$$

The second entry in the product is the product of the second row of the left-hand matrix with the column vector of variables, that is:

$$\begin{pmatrix} 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1(x) + 0(y) + 3(z) = x + 3z.$$

For another example,

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 0 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -4 \\ -1 \end{pmatrix}.$$

The third entry is

$$\begin{pmatrix} 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -4.$$

Notice that the number of entries in the rows of A must be the same as the number of entries in the column vector X for this to work: We can multiply an $n \times m$ matrix times a $p \times 1$ column vector *only if* $m = p$.

We can use this multiplication to rewrite a system of linear equations as a single matrix equation. For example, let us look at the system of equations

$$v + w + x + y + z = 1$$

$$v - w + x - y + z = 3$$

$$3v - w + 3x - y + 3z = 7.$$

Notice that when we multiply the matrix of coefficients

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 3 & -1 & 3 & -1 & 3 \end{pmatrix}$$

by the column vector of variables

$$X = \begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix}$$

we get the column vector

$$AX = \begin{pmatrix} v + w + x + y + z \\ v - w + x - y + z \\ 3v - w + 3x - y + 3z \end{pmatrix}$$

whose entries are the left-hand sides of the equations in the system. For the system of equations to be satisfied, these entries should equal the right-hand sides of the equations, or the entries in the column vector of constant terms

$$B = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}.$$

For two vectors to have the same entries, of course, means that they are equal. So our system of linear equations says the same thing as the single matrix equation

$$AX = B$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 3 & -1 & 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}.$$

Any system of linear equations can be rewritten in this way as the matrix equation

$$AX = B$$

where A is the coefficient matrix, X is the column vector of variables, and B is the column vector of constant terms.

Example: Rewrite the system of linear equations

$$x + 2y = 3$$

$$2x + 6y = 8$$

as a single matrix equation.

Solution:

$$\begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}.$$

1.2 Solving Matrix Equations

Since a matrix equation $AX = B$ (where X is a column vector of variables) is equivalent to a system of linear equations, we can use the same methods we have used on systems of linear equations to solve matrix equations. Namely:

(1.) Write down the augmented matrix $A \vdots B$.

(2.) Row-reduce to a new augmented matrix $\overline{A} \vdots \overline{B}$ in row echelon form.

(3.) Use this new matrix to write a matrix equation equivalent to the original one.

(4.) Use this new, equivalent matrix equation to find the solutions to the original equation.

For example, the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$$

can be solved by writing down the augmented matrix

$$\begin{pmatrix} 1 & 2 & \vdots & 3 \\ 2 & 6 & \vdots & 8 \end{pmatrix},$$

row-reducing it (make sure you can fill in all of the steps required to convert each of the matrices in this series to the next matrix):

$$\begin{pmatrix} 1 & 2 & \vdots & 3 \\ 0 & 2 & \vdots & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & \vdots & 3 \\ 0 & 1 & \vdots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \vdots & 1 \\ 0 & 1 & \vdots & 1 \end{pmatrix},$$

and writing the new matrix equation to correspond to this new augmented matrix,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Multiplying out the left-hand side,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we see that this system is already solved. So the solution to our original matrix equation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Here is another example. Suppose we begin with the matrix equation

$$\begin{pmatrix} 2 & 4 & 0 & 8 \\ 1 & 2 & 1 & 10 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

The augmented matrix

$$\begin{pmatrix} 2 & 4 & 0 & 8 & \vdots & 4 \\ 1 & 2 & 1 & 10 & \vdots & 3 \end{pmatrix}$$

row-reduces to the form

$$\begin{pmatrix} 1 & 2 & 0 & 4 & \vdots & 2 \\ 0 & 0 & 1 & 6 & \vdots & 1 \end{pmatrix},$$

which corresponds to a matrix equation equivalent to our original one,

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

or,

$$\begin{pmatrix} w + 2x + 4z \\ y + 6z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

To read off the solution from this equation, notice that the first row of the row-reduced matrix has a leading entry in column 1 (the column corresponding to w) and the second row has a leading entry in column 3 (the column corresponding to y.) We can separate w and y from the first and second entries in the left-hand vector by rewriting our equation as

$$\begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} 2 - 2x - 4z \\ 1 - 6z \end{pmatrix}.$$

Now, assigning parameters s and t to the variables x and z that did not correspond to leading entries, our solution is

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-2s-4t \\ s \\ 1-6t \\ t \end{pmatrix}.$$

1.3 "Dividing" By Matrices

Seeing a matrix equation written out as

$$AX = B,$$

you might be tempted to think we can solve this equation by somehow dividing both sides by A. There is sometimes a sense in which we can do this. Sometimes a matrix A has an *inverse*, another matrix A^{-1} with the property that, no matter what X is,

$$A^{-1}AX = X.$$

Then we can "divide by A" by multiplying by A^{-1}, to convert our matrix equation to

$$A^{-1}AX = A^{-1}B,$$
$$X = A^{-1}B.$$

This is analogous to dividing through both sides of a numerical equation $ax = b$ by multiplying through by $\frac{1}{a}$.

There is an important warning here. We are used to thinking that multiplying an equation through by a number always gives an equivalent equation. This is not quite true, which we also know, if we think about it; multiplying an equation $ax = b$ through by 0 gives the equation $0 = 0$, which is not equivalent to $ax = b$. However, we are used to this exception by now, and (usually) take it into account automatically.

Similarly, multiplying a matrix equation $AX = B$ by another matrix sometimes gives an equivalent equation, but not always. When multiplying a matrix equation by a matrix, there are more exceptions than just multiplication by the zero matrix (the matrix with zeroes in every entry.) For example, the matrix equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

can be rewritten, multiplying out the left hand side, as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

If we multiply this equation through by the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

we get

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

or, multiplying out both sides,

$$\begin{pmatrix} x + 2y \\ 2x + 4y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}.$$

The first equation has only one solution, the point $(x, y) = (1, 2)$, while the second has as solutions all points on the line $x + 2y = 5$; so the second equation was not equivalent to the original one.

Later we will discuss the circumstances in which we can multiply a matrix equation by a matrix A and obtain an equivalent equation. We will show that you can do this only when A has an inverse. (This is completely analagous to the situation in which you multiply both sides of an equation by numbers. You can't multiply by zero as we have already discussed, and zero has no inverse. If a number is nonzero it has an inverse and in that case it can multiply both sides of an equation without changing the nature of the solutions.) We will also see how to find out whether A has an inverse, and how to find A^{-1} if it does.

2 Solutions To Matrix Equations

2.1 Reading the solutions from the row echelon form

Remember the example of the matrix equation

$$\begin{pmatrix} 2 & 4 & 0 & 8 \\ 1 & 2 & 1 & 10 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

The augmented matrix

$$\begin{pmatrix} 2 & 4 & 0 & 8 & \vdots & 4 \\ 1 & 2 & 1 & 10 & \vdots & 3 \end{pmatrix}$$

row-reduced to the row echelon form

$$\begin{pmatrix} 1 & 2 & 0 & 4 & \vdots & 2 \\ 0 & 0 & 1 & 6 & \vdots & 1 \end{pmatrix}.$$

Here is the way we used that form to get the solutions:

The four columns of the coefficient matrix correspond to the four variables w, x, y and z. The leading entry of row 1 is in the w-column, so the first equation in the equivalent system of linear equations will determine the value of w. The leading entry of row 2 is in the y-column, so the second equation in the equivalent system of linear equations will determine the value of y. The system of linear equations is

$$w + 2x + 4z = 2$$

$$y + 6z = 1$$

so the first equation gives us

$$w = -2x - 4z + 2$$

and the second equation gives us

$$y = -6z + 1.$$

No row has a leading entry in either the x-column or the z-column, so there are no equations to determine the values of x and z; x and z can be anything, so we assign them *parameters*,

$$x = s$$

$$z = t.$$

Putting this together, we get

$$\begin{aligned} w &= -2s - 4t + 2 \\ x &= s \\ y &= -6t + 1 \\ z &= t \end{aligned}$$

as our complete solution; or in matrix form,

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s - 4t + 2 \\ s \\ -6t + 1 \\ t \end{pmatrix}.$$

Whenever we have put the coefficient matrix of a system into row echelon form, if the column corresponding to a variable does not contain the leading entry of any row, that variable will be assigned a parameter. The *dimension* of the collection of solutions (the **solution space**) is the number of parameters we need to use. So the dimension of the solution space is the number of rows without leading entries.

You may have learned at some time that "you need to have the same number of equations as unknowns." What does this mean? If we have fewer equations than variables in a system of linear equations, then there are fewer rows than columns in the coefficient matrix, so after we put the matrix into row echelon form there must be at least one column that does not contain the leading entry of any row. This means that expressing the complete solution to the system will require at least one parameter. There may be no solutions to the system, or there may be infinitely many, but there cannot be a unique solution.

Example: The coefficient matrix of a system of linear equations has the following row echelon form:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

What can we say about the complete solution to the system?

Solution: This was a system of three linear equations in three variables, let's say x, y and z; finding the solution corresponds to finding the intersection of three planes. The z-column does not contain the leading entry in any row, so we would need a parameter for z to describe the complete solution. There are two possibilities:

The system has solutions. In this case, it must have infinitely many solutions. The solution space is one-dimensional, because we need one parameter to describe it; that is, the three planes intersect in a line.

The system has no solutions; the three planes do not intersect. This would happen if the *augmented* matrix of the system had the row echelon form

$$\begin{pmatrix} 1 & 0 & 2 & \vdots & 0 \\ 0 & 1 & 3 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 1 \end{pmatrix},$$

because the equivalent system of linear equations would include the equation $0 = 1$.

We can't tell which happens because we do not know the row echelon form of the augmented matrix; we don't know anything about the column that corresponds to the constant terms of the equations. We can say that the system either has infinitely many solutions forming a one-dimensional solution space, or else no solutions at all.

2.2 Linear functions

Multiplying a column vector X by a matrix A gives a new column vector AX (possibly of a different dimension, depending on the dimensions of A). If A is an $m \times n$ matrix, then X must be an $n \times 1$ matrix, or an n-dimensional column vector, and the product AX will be an $m \times 1$ matrix, or an m-dimensional column vector.

You can think of this multiplication as defining a *function* L_A from n-dimensional column vectors to m-dimensional column vectors, given by

$$L_A(X) = AX.$$

Functions of this kind are called **linear**, and play an important role in mathematics. The main idea behind differentiation is to produce linear approximations (like the tangent line approximation) to functions.

The reason that these functions are called linear is that they satisfy the same two important linearity properties that linear differential operators satisfy: They preserve addition and multiplication by constants.

$$L_A(X+Y) = A(X+Y) = AX + AY = L_A(X) + L_A(Y)$$
$$L_A(cX) = A(cX) = c(AX) = c(L_A(X))$$

Example: Suppose

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad Y = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Then

$$A(X+Y) = A\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right) = A\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$AX + AY = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

Similarly,

$$A(4X) = A\left(4\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = A\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 20 \\ -8 \end{pmatrix}$$

$$4(AX) = 4\left(A\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = 4\left(\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = 4\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 20 \\ -8 \end{pmatrix}.$$

2.3 Homogeneous and non-homogeneous equations

You recall that a linear differential equation

$$f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y = g(x)$$

was called homogeneous if $g(x) = 0$, and non-homogeneous or inhomogeneous otherwise. We use the same terminology for systems of linear equations and for matrix equations:

A matrix equation

$$AX = B$$

is called **homogeneous** if B is the zero vector (all entries are zero). A system of linear equations is called **homogeneous** if the equivalent matrix equation is homogeneous.

Homogeneous matrix equations have some special properties:

1. The matrix equation

   $$AX = 0$$

   always has at least one solution, the zero solution

   $$X = 0.$$

   (Here 0 stands for a column vector all of whose entries are zero.)

2. If the column vectors X_1 and X_2 are two solutions to the matrix equation

   $$AX = 0$$

   then so is any linear combination of them, $aX_1 + bX_2$.

3. The complete solution to a matrix equation

   $$AX = 0$$

   is always given in the form

   $$X = t_1X_1 + t_2X_2 + \cdots + t_nX_n,$$

   where X_1, X_2, ... X_n are solutions and t_1, t_2, ... , t_n are parameters. The number of parameters depends on the dimension of the "solution space."

You can see why property (1) holds; a system of linear equations like

$$\begin{aligned}
a_1x_1 + a_2x_2 + \cdots + a_nx_n &= 0 \\
b_1x_1 + b_2x_2 + \cdots + b_nx_n &= 0 \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n &= 0 \\
\vdots \qquad\qquad & \quad \vdots
\end{aligned}$$

will always be satisfied by setting all the variables equal to zero. (This is the same reason a homogeneous linear differential equation can always be satisfied by setting $y = 0$.)

Property (2) depends on the linearity of multiplication by A. If

$$AX_1 = 0 \qquad \text{and} \qquad AX_2 = 0$$

then we have that

$$A(aX_1 + bX_2) = A(aX_1) + A(bX_2) = aAX_1 + bAX_2 = 0 + 0 = 0.$$

Property (3) also really comes from the linearity, since if we have

$$AX_1 = 0 \qquad AX_2 = 0 \qquad \cdots \qquad AX_n = 0$$

then we have that

$$A(t_1X_1 + t_2X_2 + \cdots + t_nX_n) =$$

$$t_1A(X_1) + t_2A(X_2) + \cdots + t_nA(X_n) = 0 + 0 + \cdots + 0 = 0.$$

This is the same reason that the general solution to a homogeneous linear differential equation is a linear combination of particular solutions, such as

$$y = A\cos x + B\sin x.$$

In the case of differential equations, the number of different particular solutions, or the number of constants in the general solution, depends on the order of the differential equation; one solution for a first order equation, two different solutions for a second order equation, etc. In the case of matrix equations, the number of particular solutions is the number of paramters in the general or complete solution, the dimension of the solution space.

We can also see property (3) in action by solving a matrix equation. Here's the equation:

$$\begin{pmatrix} 2 & 4 & 0 & 8 \\ 1 & 2 & 1 & 10 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The augmented matrix of this equation has the row echelon form

$$\begin{pmatrix} 1 & 2 & 0 & 4 & \vdots & 0 \\ 0 & 0 & 1 & 6 & \vdots & 0 \end{pmatrix}$$

so we can write down the complete general solution

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s - 4t \\ s \\ -6t \\ t \end{pmatrix}.$$

We can rewrite this as

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ -6 \\ 1 \end{pmatrix}.$$

The particular solutions from which we can put together this complete solution are

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -4 \\ 0 \\ -6 \\ 1 \end{pmatrix}.$$

The really nice thing we get out of this is a method for finding solutions to *non-homogeneous* systems of linear equations (or non-homogeneous matrix equations.) It works exactly the same way as solutions for linear differential equations:

If the matrix equation

$$AX = B$$

has one particular solution X_p, and the associated homogeneous equation

$$AX = 0$$

has the complete solution X_h, then the complete solution to the original non-homogeneous equation is

$$X = X_p + X_h.$$

Example:

$$\begin{pmatrix} 2 & 4 & 0 & 8 \\ 1 & 2 & 1 & 10 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

has the complete solution (which we computed earlier)

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s - 4t + 2 \\ s \\ -6t + 1 \\ t \end{pmatrix},$$

which we can rewrite as

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ -6 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

This is the sum of the solution to the associated homogeneous system, which we wrote down in the previous example,

$$X_h = s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ -6 \\ 1 \end{pmatrix},$$

and a particular solution to this inhomogeneous system

$$X_p = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Example: The homogeneous system of linear equations

$$x + y + z = 0$$

$$x - y + 2z = 0$$

$$3x - y + 5z = 0$$

has the complete solution

$$x = -\frac{3}{2}t$$

$$y = \frac{1}{2}t$$

$$z = t.$$

The non-homongeous system

$$x + y + z = 3$$

$$x - y + 2z = 2$$

$$3x - y + 5z = 7$$

has one particular solution

$$x = 1$$

$$y = 1$$

$$z = 1.$$

To get the complete solution to the non-homongeneous system

$$x + y + z = 3$$

$$x - y + 2z = 2$$

$$3x - y + 5z = 7$$

we add these together:

$$x = -\frac{3}{2}t + 1$$

$$y = \frac{1}{2}t + 1$$

$$z = t + 1.$$

Exercise 1 *Rewrite the systems of linear equations of Exercise 4 (in the last handout) as matrix equations.*

Exercise 2 *Rewrite the following matrix equations as systems of linear equations.*

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 0 & 1 \\ 9 & 8 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

Exercise 3 *Carry out the following matrix multiplications, or explain why they cannot be carried out.*

$$\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Exercise 4 *Solve the following matrix equations using row-reduction.*

$$\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 4 & 2 & 8 \end{pmatrix} \begin{pmatrix} v \\ w \\ x \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Exercise 5 *What does it say about the set of solutions to a system of linear equations if its augmented matrix, when put into row-reduced form:*

(a.) Has a row whose leading entry is in the last column (the column corresponding to the constant terms)?

(b.) Has all zeroes in the last column?

(c.) Has a column, other than the last column, in which no row has a leading entry?

(d.) Has rows with leading entries in every column except the last one?

Exercise 6 *Write down the associated homogeneous matrix equations for the matrix equations in exercise 4. Now write down the complete solution to each of these homogeneous matrix equations.*

Exercise 7 *The matrix equation*

$$\begin{pmatrix} 2 & 1 & -1 \\ 4 & 2 & 8 \end{pmatrix} \begin{pmatrix} v \\ w \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 26 \end{pmatrix}$$

has one solution given by

$$\begin{pmatrix} v \\ w \\ x \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Give the complete solution to this matrix equation.