Math 8
Fall 2015

Preliminary Homework
Due Friday, September 18

Note: Preliminary homework is always graded credit or no credit. **You get full credit for completing the assignment, whether or not your answers are correct.** The purpose of preliminary homework is to start you thinking about the topic of the next class.

You may use your preliminary homework in activities with your classmates. You should be sure to think about these questions so you will be prepared.

Preliminary homework is always due at the *beginning* of class.

In class, we discussed using Taylor polynomials to approximate functions. If we want to use $P_n(x)$ as an approximation to $f(x)$, we may well want to know just how good that approximation is; that is, how large the error

$$\text{ERROR} = |R_n(x)| = |f(x) - P_n(x)|$$

is. We might also want to be able to choose a value of n that will make the error smaller than some predetermined tolerance; for example, we might want $P_n(x)$ to approximate $f(x)$ to within 2 decimal places, or possibly to within as many decimal places as display on the calculator we are designing.

A number B is a bound on the error if the error is guaranteed to be no bigger than B. These exercises lead up to a formula for a bound on the error in using $P_1(x)$, the tangent line approximation, as an approximation to $f(x)$.

Recall Rolle's Theorem: If $f'(x)$ is defined and continuous on the interval $[a, b]$, and $f(a) = f(b)$, then there is a point d in the interval (a, b) such that

$$f'(d) = 0.$$

Assume that $f'(x)$ and $f''(x)$ are defined and continuous everywhere, so Rolle's theorem will apply to the functions g and g' defined below.

The first Taylor polynomial for $f(x)$ at the point a is

$$P_1(x) = f(a) + f'(a)(x - a).$$

Suppose that $c \neq a$, and we want to use $P_1(c)$ as an approximation for $f(c)$. The following steps will lead to a bound on the error.

First we define a new function by

$$g(x) = f(x) - P_1(x) - \frac{(x-a)^2}{(c-a)^2}\Big(f(c) - P_1(c)\Big).$$

1. Find expressions for
   (a) $g'(x)$ and
   (b) $g''(x)$.

   (Remember that a and c are both constants.)

2. Show the following. (Rolle's Theorem may be helpful with some parts.)
   (a) $g(c) = 0$, $g(a) = 0$ and $g'(a) = 0$.
   (b) There is a point d between a and c for which $g'(d) = 0$.
   (c) There is a point w between a and d for which $g''(w) = 0$.

3. Use (1b) and (2c) to find an expression for $R_1(c)$ in terms of $f''(w)$, where w is the same w as in (2c). Recall that $R_1(c) = f(c) - P_1(c)$.

4. Of course, this doesn't tell us what $R_1(c)$ is, because we don't know what w is. But suppose M is a number such that $|f''(x)| \leq M$ for every x in the interval $[a, c]$ (including, of course, w). Then

   $|R_1(c)| \leq$ ______________________________________.

5. According to your answer to part (4), if we know the second derivative $f''(x)$ is small on the interval between a and c (so M is small), then we know the tangent line approximation to $f(x)$ near the point a is fairly close to the actual function $f(x)$ at the point c. Why does this make sense?

6. Suppose $P_1(x)$ is the first Taylor polynomial for $f(x) = e^x$ at the point $a = 0$. Use the formula you found in part (4) to find a bound on the error in using $P_1(-.01)$ as an approximation to $e^{-.01}$.

   Hint: When you are trying to find the number M in the formula, remember that if $x \leq 0$, then $e^x \leq e^0 = 1$.