MATH 8 CLASS 9 NOTES, 10/11/2010

Last time, we were interested in finding power series expansions for various functions. We started by looking at the geometric series (which is also a power series)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

which is valid for $|x| < 1$. From this example, we found other examples of functions we could find power series expansions for by either

(1) Substituting an expression bx^d for x, where b is a real number, d a positive integer,
(2) Multiplying a function we know a power series expansion for by a power of x,
(3) Differentiating a power series term-by-term,
(4) and integrating a power series term-by-term.

For example, we found that we could find a power series expansion for arctan by integrating its derivative, $1/(1+x^2)$, and obtained

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

1. More on integrating power series

One useful fact about power series is that we can also obtain expressions for integrals of functions we would not usually know how to integrate.

Examples.

- Find the integral of $\arctan x$, expressed as a power series. We integrate the power series expansion for $\arctan x$ term by term:

$$\int \arctan x \, dx = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{2n(2n-1)}.$$

(We suppress the $+C$ for convenience.) This answer is not totally satisfactory, since we do not have an expression for this integral in terms of functions we know, but in practice this answer can be quite useful. For example, this answer can allow us to numerically estimate definite integrals of $\arctan x$ to high precision.

- Estimate

$$\int_0^1 \arctan x \, dx$$

to within an error of $0.01 = 1/100$. Using the expression for the indefinite integral of $\arctan x$ in the previous example, we find that

$$\int_0^1 \arctan x \, dx = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \dots \Big|_0^1 = \frac{1}{2} - \frac{1}{12} + \frac{1}{30} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n(2n-1)}.$$

Notice that we are in the fortunate situation where we have an alternating series which passes the alternating series test! In this case, we can estimate our series by taking the first n terms of the series, and the error is bounded in size by the first term we omit. We want error $\leq 1/100$, so this suggests that we should take the approximation

$$\frac{1}{2} - \frac{1}{12} + \frac{1}{30} - \frac{1}{56} + \frac{1}{90}$$

for the value of the definite integral we wanted to calculate. The first omitted term is $1/(12 \cdot 11) = 1/132 < 1/100$, so the error is bounded by 0.01, as desired.

2. Power series for general functions: Taylor series

Let $f(x)$ be a function which has derivatives of every order. For example, most functions we are familiar with, such as $\sin x, \cos x, e^x$ satisfy this property. We want to find out how to approximate $f(x)$ using polynomials, and perhaps even find a power series expansion for $f(x)$.

For example, let $f(x) = e^x$. We already saw how to obtain the power series for this function (at least around $x = 0$) by being clever about the fact that e^x is its own derivative, but let us consider another way to find the power series for e^x. We start by considering the simpler problem of approximating e^x using polynomials. More concretely, we will want to approximate $f(x) = e^x$ at the point $a = 0$ using polynomials of progressively higher degrees.

If we are not ambitious at all, we can try to approximate $f(x)$ at $x = 0$ using a constant function. The best approximation is obviously the value of $f(0) = 1$, so $f(x) = 1$ is the best 0th degree polynomial which approximates $f(x) = e^x$ at $x = 0$. If we are slightly more ambitious, we can ask which line is the best approximation to $f(x)$. Let this line have equation $y = c_0 + c_1 x$. Then this line should definitely pass through the point $(0, 1)$, so this means that $1 = c_0$. Also, we should expect the slope of this line to be equal to the slope of e^x at $x = 0$: that is, this line and e^x should have the same first derivative at $x = 0$. This imposes the requirement that $c_1 = 1$ as well, so we find that $f(x) = x + 1$ is the line which best approximates e^x at $x = 0$.

We can continue this procedure, and ask which quadratic polynomial $f(x) = c_0 + c_1 x + c_2 x^2$ best approximates e^x at $x = 0$. Again, we want this quadratic to pass through $(0, 1)$ and have the same slope as e^x at $x = 0$. One checks that this imposes the conditions $c_0 = 1, c_1 = 1$. However, we would also like e^x and our quadratic to have the same second derivative at $x = 0$. This imposes the condition $1 = e^0 = 2c_2$, or $c_2 = 1/2$. Therefore, $f(x) = 1 + x + x^2/2$ is the best quadratic approximation to $f(x) = e^x$ at $x = 0$.

In general, suppose we want to know which nth degree polynomial $c_0 + c_1 + \ldots + c_n x^n$ best approximates e^x at $x = 0$. We want the 0th, 1st, 2nd, $\ldots$, nth derivatives of this polynomial and e^x to all match at $x = 0$. Consider the kth derivative, where $0 \leq k \leq n$. Then the requirement that the kth derivative of the two functions match imposes the condition $e^0 = k! c_k$, because the left hand side is the value of the kth derivative of e^x (which is still e^x), while the right hand side is the value of the kth derivative of our polynomial at $x = 0$.

What this means is that the best nth degree polynomial which approximates e^x is given by

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

This suggests that the power series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^n}{n!}$$

is equal to e^x, which is what we expect since this is the expression we obtained in the previous class.

For a general function $f(x)$, which has derivatives of all orders, IF it is true that $f(x)$ is equal to a power series $\sum c_n(x-a)^n$ near $x=a$, then we can find the coefficients c_n by an argument very similar to the one above. We equate the nth derivatives of $f(x)$ at $x=a$ and the power series $\sum c_n(x-a)^n$ to find the condition

$$f^{(n)}(a) = n!c_n \Leftrightarrow c_n = \frac{f^{(n)}(a)}{n!}$$

This means that if we know $f(x)$ is represented by a power series $\sum c_n(x-a)^n$ near $x=a$, that power series must be the series

$$f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A power series of this form is called the <u>Taylor series</u> for $f(x)$ at the point $x=a$. In the special case where $a=0$, we sometimes call the resulting series a <u>Maclaurin series</u>.

Example. Find the Maclaurin series for $f(x) = \cos x$, and find this series' interval of convergence. The Maclaurin series is the Taylor series for $f(x)$ at $x=0$, so what we need to do is evaluate the various derivatives of $\cos x$ at $x=0$. We find that $f'(x) = -\sin x, f''(x) = -\cos x, f'''(x) = \sin x, f^{(4)}(x) = \cos x$, so the derivatives of $f(x)$ repeat after four derivatives. This means that $f(0) = 1, f'(0) = 0, f''(0) = -1, f'''(0) = 0, f^{(4)}(0) = 1$, etc., and the Maclaurin series for $\cos x$ is given by

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

We can determine the interval of convergence of this power series in the usual way, by applying the ratio test. The limit of the ratio $|a_{n+1}/a_n|$ of the above power series is

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^2}{(2n+2)(2n+1)}\right| = 0$$

regardless of the value of x, so the ratio test tells us that this power series converges for all real x.

3. The error term for Taylor series

We still need to address the question of whether the Taylor series, at the point $x=a$, of a function $f(x)$ is equal to the function $f(x)$ itself near $x=a$. While intuition may suggest that this should always be true, it turns out that there are functions which do not equal their Taylor series at any point except $x=a$! (Check question 12.10.70(?), pg. 783 for an example.) Nevertheless, these cases turn out to be rather isolated, and the theorem which we will develop in this section allows us to determine when the Taylor series of a function is equal to the function itself.

Let $f(x)$ be a function, and let $\sum f^{(n)}(a)(x-a)^n/n!$ be its Taylor series at the point $x=a$. Let T_n be the polynomial defined by taking the sum of the first $n+1$ terms of the Taylor series:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}}{n!}(x-a)^n$$

In other words, $T_n(x)$ is the best approximation to $f(x)$, near $x = a$, amongst all polynomials of degree n. Let $R_n(x) = f(x) - T_n(x)$; that is, $R_n(x)$ is the remainder of the approximation of $T_n(x)$ to $f(x)$. The idea is that we want to show $R_n(x) \to 0$ as $n \to \infty$ for all x near a. The following theorem, which we do not prove, provides the principal means of showing this, by estimating the error of $T_n(x)$.

Theorem. (Taylor's Inequality, or Taylor Remainder Theorem) With the notation as above, consider all x satisfying $|x - a| < d$, for some d. Let M be a number such that $|f^{(n+1)}(x)| < M$ for all x satisfying $|x - a| < d$. Then for all x with $|x - a| < d$, the remainder $R_n(x)$ satisfies the following inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

The gist of this theorem is that if we can bound the $n + 1$th derivative of $f(x)$ on the interval for which we want to approximate $f(x)$, then we can bound the remainder of $f(x)$ when approximated by $T_n(x)$. Let us see how this allows us to show that a function equals its Taylor series, and how this theorem lets us numerically approximate various functions to specified degrees of accuracy.

Examples.

- Estimate e to an accuracy of $0.001 = 10^{-3}$. We use the Taylor series for e^x, centered around $x = 0$, to estimate this number. If we take the Taylor series for e^x and plug in $x = 1$, we find

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

  If we take terms up to $1/n!$ as an approximation, the Taylor remainder theorem tells us that the remainder $R_n(1)$ is bounded by

$$|R_n(1)| \leq \frac{M}{(n+1)!}1^{n+1}$$

  where M is an upper bound for e^t on the interval $|t| < 1.001$, say. In particular, we can take $M = e^{1.001}$, and using the fact that $M < 3$, this bound on the remainder becomes

$$|R_n(1)| \leq \frac{3}{(n+1)!}$$

  We want an error less than 10^{-3}. Therefore, we might want to take $n = 6$, since $(6+1)! = 7! = 5040$. The number $1 + 1/1 + 1/2! + 1/3! + 1/4! + 1/5! + 1/6!$ is thus our approximation to e.

4. Some other examples

We conclude with a few more examples which illustrate slight modifications of the techniques presented here. In the next class we will consider more examples of Taylor and Maclaurin series.

Examples.

- Calculate the Maclaurin series for x^2e^x. At first glance, one might want to calculate the derivatives of x^2e^x and evaluate them at $x = 0$ to determine the Maclaurin series for this function. However, this takes quite a bit of work and there is actually a much simpler way to solve the problem. Notice that we already know the Maclaurin series for e^x, and we are simply multiplying e^x by x^2. Therefore, one has

$$x^2e^x = x^2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) = x^2 + \frac{x^3}{1!} + \frac{x^4}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$$

  This is an application of the methods we learned in the previous class, where we could easily find power series expansions for functions which were obtained by simpler functions we already knew power series expansions for by multiplying by some power of x.
- Calculate the Maclaurin series for e^{-x^2}. Again, if one tries to take derivatives and evaluate at $x = 0$, one gets a crazy mess. It is much simpler to take the Maclaurin series for e^x, and substitute $-x^2$ for x everywhere:

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

- Find the 100th derivative of $f(x) = e^{-x^2}$ evaluated at $x = 0$. On the surface, this problem seems basically impossible, at least without a calculator or computer. Taking derivatives of e^{-x^2} quickly becomes unmanageable because you get more and more terms with each derivative. However, the question is phrased in such a way that lets us take advantage of the fact that we know the Maclaurin series for e^{-x^2}, calculated in the previous example.

  The 100th derivative of e^{-x^2} at $x = 0$ has something to do with the term corresponding to x^{100} in the Maclaurin series for e^{-x^2}. In particular, this term is equal to

$$\frac{f^{(100)}(0)}{100!}x^{100}$$

  On the other hand, we know that the 100th term of this Maclaurin series is $(-1)^{50}x^{100}/50!$. This means that $f^{(100)}(0)/100! = 1/50!$, or $f^{(100)}(0) = 100!/50!$.
- Calculate the Taylor series of $f(x) = e^{2x}$ around the point $a = 1$. To solve this problem, we need to be able to calculate the value of $f^{(n)}(1)$ for all $n \geq 0$. Notice that $f'(x) = 2e^{2x}, f''(x) = 4e^{2x}, \dots$, so $f^{(n)}(x) = 2^n e^{2x}$. This means $f^{(n)}(1) = 2^n e^2$. We then use the formula for a Taylor series to obtain the answer:

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots = e^2+2e^2(x-1)+\frac{4e^2}{2!}(x-1)^2+\frac{8e^2}{3!}(x-1)^3+\dots = \sum_{n=0}^{\infty}\frac{2^n e^2}{n!}(x-1)^n$$

  Unlike the previous examples, we cannot simply replace x with $2x$ in the Maclaurin series for e^x to solve this problem, because the problem asks for a Taylor series around $x = 1$, not $x = 0$.