MATH 8 CLASS 5 NOTES, 10/1/2010

We are still interested in the question of determining whether various series converge or diverge. Our arsenal of tests include testing geometric series, the nth term test, the integral test (of which p-series is a special case), and the limit and direct comparison tests. However, most of these tests require that the terms of a series be positive. What happens if a series has many negative terms?

1. The Alternating Series Test

An alternating series is a series $\sum a_n$ in which the a_n flip signs. For example,

$$\sum_{n=1}^{\infty}(-1)^n = -1+1-1+1-1+1-\dots$$

is an alternating series. We require that consecutive terms always have different signs, so a series like $-1+1+1-1+1+1-1-1+\dots$ would not, strictly speaking, be considered an alternating series.

We are interested in finding a way of determining when an alternating series converges. First, notice that the nth term test for divergence implies that if an alternating series converges, then its nth term must converge to 0. As it turns out, this is not enough to obtain convergence, but the addition of the condition that the absolute values $|a_n|$ form a decreasing sequence turns out to be enough to show convergence of an alternating series. We have the following:

Theorem. (The Alternating Series test). Let $\sum_{n=1}^{\infty}(-1)^n a_n$ be an alternating series, where all the $a_n \geq 0$. Then this series converges if $\lim_{n\to\infty} a_n = 0$ and a_n is a decreasing sequence.

The theorem is actually true with slightly less restrictive conditions. For example, as written, the sign of the first term is negative, and of course the theorem would still be true if the first term were positive (as long as the terms kept flipping signs). Also, that the index starts at $n = 1$ is immaterial; we could start with any other index, as long as consecutive terms have different signs. This test can only be used to check for convergence – for example, if we had an alternating series which did not satisfy the hypotheses of this test, it might still be possible for the series to converge. Never say that a series diverges because of the alternating series test. On the other hand, if you are unable to use the alternating series test because $\lim_{n\to\infty} a_n \neq 0$, then you can conclude divergence by the nth term test.

Examples.

- Consider the series $\sum_{n=1}^{\infty}(-1)^{n+1}/n = 1-1/2+1/3-1/4+\dots$. This is sometimes called the alternating harmonic series, because of the fact that it looks like the harmonic series, except with alternating signs. This series converges, because of the alternating series test. First, notice that the signs do indeed flip for each pair of consecutive terms. Secondly, the absolute values of the terms of the series form a decreasing sequence ($|a_n| = 1/n$), and finally, the limit of those absolute values is indeed 0. This is in contrast to the fact that the harmonic series diverges. Notice that even though we know that this series converges, we cannot calculate its limit yet.

- Consider the more general series $\sum_{n=1}^{\infty}(-1)^{n+1}/n^p = 1 - 1/2^p + 1/3^p - 1/4^p + \ldots$, where $p > 0$ is some real number. Then this series also converges by the alternating series test, because the signs of consecutive terms are different, the absolute values of those terms form a decreasing sequence, and have limit 0. These series are often called *alternating p-series*, for the obvious reason.
- Sometimes the sign flips can be cleverly written in a series. Consider the series $\sum_{n=0}^{\infty}\cos(n\pi)/(n+1)^2$. Although on the surface this might not look like an alternating series, notice that $\cos(n\pi) = (-1)^n$. Therefore, this really is an alternating series. It converges by the alternating series test, since $1/(n+1)^2$ is a decreasing sequence which converges to 0.
- Consider the series $\sum_{n=1}^{100} n! + \sum_{n=101}^{\infty}(-1)^n(n+1)/(n^2+1) = 1! + 2! + \ldots + 100! + \sum_{n=101}^{\infty}(-1)^n(n+1)/(n^2+1)$. Strictly speaking, this is not an alternating series, since the first hundred terms do not change signs, but after that we have a series which is alternating, and satisfies the hypotheses of the alternating series test. Therefore, if we ignore the first hundred terms, we have a series which converges. Adding back in those first hundred terms does not affect the convergence or divergence of the series (although it obviously affects the value of the series), so this series also converges.
- Does the series $\sum_{n=1}^{\infty}(-1)^n\sqrt{n+3}/\sqrt{4n+3}$ converge or diverge? This is an alternating series. Notice that the nth term of the sequence does not converge to 0: instead, as n gets large, the nth term oscillates between values close to $-1/2$ and $1/2$, because $\sqrt{n+3}/\sqrt{4n+3}$ has limit $1/2$. Therefore, this series diverges by the nth term test. (DO NOT say that the series diverges because of the alternating series test.)
- Does the series $\sum_{n=0}^{\infty}(-1)^n 2^n/n!$ converge or diverge? Again, this is an alternating series (that the index starts with $n = 0$ is irrelevant, as long as consecutive terms have different signs). Let's check that this series passes the alternating series test. First, we claim that $a_n = 2^n/n!$ is a decreasing sequence, at least when $n \geq 3$. Consider the ratio a_{n+1}/a_n; we want to show that this is < 1. Since $a_{n+1}/a_n = 2/(n+1)$, and this is obviously < 1 when $n \geq 3$.

  Next, we check that $\lim_{n\to\infty} 2^n/n! = 0$. This can be done using the squeeze theorem; for instance,

  $$\frac{2^n}{n!} = \frac{2 \cdot 2 \cdots 2 \cdot 2}{n \cdot (n-1) \cdots (2) \cdot 1} = \frac{2}{n}\frac{2}{n-1}\cdots\frac{2}{2}\frac{2}{1}.$$

  Notice that every term in the product on the right except the last two are < 1. So we have the inequality

  $$\frac{2}{n}\frac{2}{n-1}\cdots\frac{2}{2}\frac{2}{1} < \frac{4}{n}.$$

  Since $4/n \to 0$ as $n \to \infty$, and $0 \leq a_n$ is obvious, the squeeze theorem says that $2^n/n! \to 0$. Notice that the exact same argument works for any exponential a^n, not just for $a = 2$.

  In any case, this series passes the alternating series test, so it converges.

2. Error estimates for the alternating series test

Suppose we have applied the alternating series test to a series $\sum(-1)^n a_n$ to verify that the series converges. However, even if we use the alternating series test to determine convergence, we have no way of actually knowing what value the series converges to. In general, it is difficult to exactly determine the value of an alternating series. However, we can try to approximate the value of the series.

The most direct way to approximate the value of a convergent series is to sum lots of terms; that is, use the partial sums s_n, for n large, as an estimate for the actual value of the series. However, for this to be useful, we need to have a way of estimating the error of the estimate: that is, the difference between the actual value of the series (which we do not know!) and the estimate we use.

In the case of an alternating series, we have a very simple upper bound for the error:

Theorem. (Error estimate for alternating series test) Suppose the series $\sum(-1)^n a_n$ converges by the alternating series test. Let s be the value of the series, and let $s_n = \sum_{i=1}^n (-1)^i a_i$ be the nth partial sum. Then

$$|s_n - s| < |a_{n+1}|$$

In other words, if we estimate such a series by using the first n terms of the series, the error is always less than the size of the first omitted term in the estimate. Let's look at some examples:

Examples.

- Consider the alternating harmonic series $\sum_{n=1}^\infty (-1)^{n+1}/n$. This series converges and has some value. If we use the first 100 terms of the series to estimate the value of the series, then the error is less than or equal to 1/101, which is the first term that isn't used. If we use the first million terms, then the error is less than or equal to 1/1000001. Notice that this series is very hard to estimate – obtaining accuracy to N decimal places takes about 10^N terms.
- Consider the alternating series $\sum_{n=1}^\infty (-1)^{n+1}/(2n-1)! = 1/1 - 1/3! + 1/5! - 1/7! + \ldots$. (One quickly checks this passes the alternating series test.) If we use the first four terms of this series to estimate the value of the series, the error of this estimate is less than $1/9!$. In contrast to the previous example, $9! = 362880$, so the convergence of this series to its value is very rapid, in the sense that we only have to take a few terms to get a reasonably accurate estimate. For instance, $s_4 = 0.841468254...$ is the value when we use four terms in this series, while the actual value of the series is 0.841470984.... So we get four digits of accuracy with four terms in this example, while in the previous example (alternating harmonic series) we would need something like $10,000$ terms.
- A common type of question you might see is as follows: given a series $\sum(-1)^n a_n$ which passes the alternating series test, what is the least number of terms you can use to approximate the series to within some given amount of error? For example, given the series $\sum_{n=1}^\infty (-1)^n/(n^2+2)$, one checks this passes the alternating series test. What is the least number of terms you need to ensure that the error is less than $1/10^6$? The 1000th term is $(-1)^{1000}/(10^6+2)$, which is just barely less than $1/10^6$. Since $1/(999^2+2) > 1/10^6$, we see that we need at least 999 terms to ensure that the error is less than $1/10^6$. (Notice that it is possible for $|s_{999} - s|$ to be less than $1/10^6$, but we don't actually know if this is true using the error estimate for alternating series. So we are looking for the smallest n for which the error estimate for alternating series can guarantee that $|s_n - s| < 1/10^6$.)

3. Conditional and absolute convergence

We saw that the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ converges, while the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges. We can obtain the harmonic series from the alternating harmonic series by taking the absolute value of each term of the series. This type of situation is important enough that we give it a name:

Definition. Suppose $\sum a_n$ converges but $\sum |a_n|$ diverges. Then we call $\sum a_n$ a <u>conditionally convergent</u> series. If instead both $\sum a_n$ and $\sum |a_n|$ converge, then we call $\sum a_n$ an <u>absolutely convergent</u> series.

Notice that we do not require $\sum a_n$ to be an alternating series in this definition.

Examples.

- The alternating harmonic series is a conditionally convergent series. More generally, the alternating p-series, for $0 < p < 1$, is also conditionally convergent, since we said earlier that the alternating p-series converges, while the p-series $\sum 1/n^p$ diverges, when $0 < p < 1$.
- The alternating p-series for $p > 1$ is absolutely convergent, because the p-series $\sum 1/n^p$ is convergent for $p > 1$.
- Is the series $\sum_{n=1}^{\infty}(-1)^n 3/\sqrt{n}$ divergent, conditionally convergent, or absolutely convergent? Notice that this is just 3 times the alternating p-series for $p = 1/2$. Since multiplying by a constant does not affect convergence or divergence of a series, this series is conditionally convergent.
- Is the series $\sum_{n=1}^{\infty}(-1)^n 3^{n+1}/5^n$ divergent, conditionally convergent, or absolutely convergent? This series is a geometric series with $r = -3/5$, so it converges. The series obtained by taking absolute values of each term is $\sum 3^{n+1}/5^n$, which is geometric with ratio $r = 3/5$, so it also converges, and the original series is absolutely convergent.

The examples we have seen strongly suggest that if $\sum |a_n|$ converges, then $\sum a_n$ also converges. This is in fact true. Even though this may seem intuitively obvious (or maybe not), it is a little bit tricky to actually prove this statement, and we will not do so here. However, this fact does allow us to add to the list of series that we can show converge:

Example.

- Does the series $\sum_{n=1}^{\infty} \sin n/n^2$ converge or diverge? Notice that the sign oscillation in this problem is irregular, because the sign of $\sin n$ does not change with any sort of regular pattern. In particular, we cannot use the alternating series test. Nevertheless, if we take absolute values we obtain the series $\sum_{n=1}^{\infty} |\sin n|/n^2$, and because $|\sin n| \le 1$, we can use direct comparison to compare this series with $\sum 1/n^2$. This is convergent, being a p-series with $p > 1$, so $\sum |\sin n|/n^2$ converges, and therefore the original series $\sum \sin n/n^2$ converges as well.