MATH 8 CLASS 3 NOTES, 9/27/2010

1. The Integral Test

Recall what the nth term test for divergence says: given a series $\sum a_n$ (the beginning index does not matter), if $\lim_{n\to\infty} \neq 0$ or if $\{a_n\}$ diverges, then $\sum a_n$ also diverges. However, if $\lim_{n\to\infty} = 0$, the nth term test cannot be used, and we need some other methods to determine whether a series diverges or converges. However, right now, the only other general method we know is to identify the series as a geometric series and then examine the ratio. Unfortunately, it is rather rare (both in real life and in math problems) to encounter a geometric series.

So we will begin to create a set of tools which will help us determine the convergence or divergence of various series. We begin with a tool which will help us determine whether the series $\sum_{n=1}^{\infty} 1/n$ diverges or converges.

This series is known as the *harmonic series*. We can numerically calculate a few partial sums; for instance, $s_{10} \approx 2.929, s_{100} \approx 5.187, s_{1000} \approx 7.485$, and $s_{10000} \approx 9.788$. So perhaps the harmonic series is diverging, but if it is doing so, it is growing very slowly. How can we actually show whether this series is diverging or not?

Notice that the terms of the harmonic series, $1/n$, are just the values of the function $f(x) = 1/x$ at positive integers. Recall that $\int_a^b 1/x\, dx$ is the area under the curve of the function $1/x$, from $x = a$ to $x = b$. Now let us represent the partial sum

$$s_n = \sum_{i=1}^{n} \frac{1}{i}$$

of the harmonic series as the sum of the areas of several rectangles drawn in the following way: (if these notes were ideal I'd have a picture here, but I'm going to draw it on the chalkboard) on the interval $[i, i+1]$, we draw a rectangle of height $1/i$. This rectangle has area $1/i$. So if we draw these rectangles from $[1, 2]$ all the way to $[n, n+1]$, we obtain n rectangles whose areas sum to s_n.

The great insight is that region these rectangles cover contain the area under the curve of the graph of $f(x) = 1/x$, from $x = 1$ to $x = n + 1$. That is, we have the following inequality:

$$\int_1^{n+1} \frac{1}{x}\, dx \leq \sum_{i=1}^{n} \frac{1}{i}.$$

However, we know how to calculate the left hand side! It is simply $\ln(n + 1)$. So we obtain the inequality

$$\ln(n + 1) \leq s_n.$$

So what happens as $n \to \infty$? Since $\ln(n + 1) \to \infty$ as $n \to \infty$, we must also have $s_n \to \infty$. Therefore, the harmonic series diverges.

Let's try this idea on the series $\sum_{n=1}^{\infty} 1/n^2$ instead. Again, we can draw rectangles over the interval $[i, i + 1]$ of height $1/i^2$. These rectangles cover the area under the curve of the function $f(x) = 1/x^2$. However, this time, we see that $\int 1/x^2\, dx = -1/x$, so

$$\int_1^{n+1} 1/x^2\,dx = \frac{-1}{x}\Big|_1^{n+1} = 1 - \frac{1}{n+1},$$

and as $n \to \infty$ this goes to 1, not ∞. So the previous idea of using this integral to show that the series $\sum_{n=1}^{\infty} 1/n^2$ diverges isn't going to work. However, we can slightly alter the idea to use the fact that this integral converges as $n \to \infty$ for us, to show that the series in question converges!

All we have to do is draw a rectangle of height $1/i^2$ over the interval $[i-1, i]$, instead of $[i, i+1]$ this time. We might omit the rectangle corresponding to $i = 1$, but we can still draw the rest. This time, notice that these rectangles are all inside the area under the curve of the graph of $f(x) = 1/x^2$ from $x = 1$ to $x = n$. So we obtain the inequalities

$$\sum_{i=2}^{n} \frac{1}{i^2} \le \int_1^n \frac{1}{x^2}\,dx = 1 - \frac{1}{n}.$$

Why does this help us? Since the right hand side is bounded from above by 1, all the partial sums $\sum_{i=2}^{n} 1/i^2$ are also bounded from above by 1. (It doesn't matter that we discard the first term; we could add that back in, and then all partial sums are bounded from above by 2, which will not change what follows.) But these partial sums are also a monotonic increasing sequence (since all the terms $1/i^2$ are positive), so we can use the monotone convergence theorem to conclude that $\sum_{i=1}^{\infty} 1/i^2$ converges.

So what were the key ideas in these two examples? We tried to determine convergence or divergence by relating the partial sums of the series we were interested in to areas under curves of functions which we knew how to integrate. In particular, we wanted to express these partial sums as areas of rectangles which either contained or were completely contained by the area under the graph of a function which we could integrate. This function should give the terms of the series when evaluated at integers, as well.

Without a lot of additional modification from the two examples above, we can obtain the following test for convergence or divergence:

The integral test. Let $f(x)$ be a continuous, monotone decreasing, positive function on $[1, \infty)$. Suppose $a_n = f(n)$ for all positive integers n. Then the series $\sum a_n$ converges if and only if the integral

$$\int_1^{\infty} f(x)\,dx$$

converges. (When we say that this integral converges, we really mean

$$\lim_{n\to\infty} \int_1^n f(x)\,dx$$

converges to a limit, as opposed to diverging to infinity. Such an integral is sometimes known as an *improper integral.*) Similarly, $\sum a_n$ diverges if and only if the integral above diverges (to infinity).

If you use this test, it is essential that the function you choose be monotone decreasing and positive. Actually, the fact that the function is monotone starting at $x = 1$ is not essential, but it is essential that it *eventually* becomes monotone decreasing. For instance, perhaps a series starts as $1 + 10 + 3.5 - 4 + \ldots$. Although this certainly does not give a monotone decreasing function, if the rest of the terms are of the series have the form $1/n^2$, say, then you can apply the integral test just starting at the 5th term, since from that point on the terms of the series are the integral values of a continuous monotone decreasing positive function. This is just one interpretation of the fact that the convergence or divergence of a series (and/or sequence) is not impacted by some given finite number of terms.

The integral test looks pretty versatile. But when will you probably not be able to use it? Well, first of all, you can only apply the integral test if the series you are interested in has all positive terms. If a finite number of terms are negative, that's okay, since you can discard them without impacting the convergence of the series, but if your series has an infinite number of negative terms, like $\sum(-1)^n/n$, the integral test won't help you.

Your terms also need to be decreasing. Again, if some finite number of terms are problematic, you can throw them out, but for some series, such as $\sum(2+(-1)^n)/n$, the terms jump up and down. (The first few terms are $1+3/2+1/3+3/4+1/5+3/6+\ldots$, so you can see that this is not a monotone decreasing sequence.) So the integral test isn't directly applicable here either.

You may also be in a situation where the terms simply can't be easily interpreted as the values of a real-valued function at the positive integers. For example, it is hard to directly use the integral test on a series like $\sum 1/n!$, since you don't know how to write down a real valued function equal to $1/n!$ at the positive integers.

If you think a series can be tested using the integral test, be sure to check that you can find a real-valued function which gives the terms of the series when evaluated at positive integers, and that the function is continuous and monotone decreasing for all of $[1,\infty)$. Furthermore, this function should be one you can integrate without too much trouble. If all these conditions are passed, then the integral test will be able to tell you if your series converges or diverges. And when you actually use the integral test on an exam or assignment, be sure to say that you are doing so and to check that all the conditions required for using the integral test are satisfied.

The first and one of the most useful applications of the integral test generalizes the two examples above:

The p-series. Let $p>0$ be some fixed positive real number. The series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

is called the *p-series.* If $p\le 1$, then this series diverges, while if $p>1$, the series converges.

To prove this statement, just apply the integral test to the above series. We've already seen what happens when $p=1$; if $p\neq 1$, we want to determine the convergence of the integral

$$\int_1^{\infty}\frac{1}{x^p}\,dx=\frac{x^{-p+1}}{-p+1}\Big|_1^{\infty}.$$

If $p<1$, then the expression on the right hand side is some positive power of x, so diverges. On the other hand, if $p>1$, then the expression on the right hand side is a negative power of x, so the right hand side equals 1, and thus converges.

2. Approximating errors in the integral test

When $p>1$, the p-series is perhaps the first example of a series which we can show converges without actually being able to calculate the value of the series. For instance, when $p=2$, what's the value of that series? As a matter of fact, calculating these exact values is a very difficult problem which we will not be able to cover in this class. Notice that the value of the series definitely is not equal to the value of the integral

$$\int_1^\infty \frac{1}{x^p}\,dx,$$

since this represents the area of a region which is strictly larger than the area of the rectangles corresponding to the p-series (after discarding the first term).

Nevertheless, perhaps we are just interested in approximating the value of a p-series. For instance, when $p = 2$, we know that the series $\sum_{i=1}^{\infty} 1/i^2$ converges to some value. Since the partial sums are monotone increasing, it is clear that we can approximate this value by simply adding the first few terms of the series, and the more terms we add, the better we approximate this value.

However, since we don't know what the value of this series is to begin with (let us call it s), for these approximations to be useful we need to have a way of estimating how far off from the actual value we are when we use partial sums as estimates for s. For example, for $p = 2$, $s_{10} = 1/1^2 + \ldots + 1/10^2 \approx 1.550$, while $s_{100} \approx 1.635$. This is all well and good, but without being able to estimate the error in these approximations, we aren't getting much useful information. For example, if the actual value of s were 100, say, these would be bad approximations. But we'll see how to show that s_{100} can't be more than $1/100$ away from the value of s.

So suppose we have a series which we used the integral test on to show that it converges, by comparing it to the indefinite integral of a function $f(x)$. What's the key idea in estimating the error of these approximations? Suppose we are using $s_n = a_1 + a_2 + \ldots + a_n$ in estimating the value of s. We call $s - s_n = R_n$ the *error* of the nth partial sum (the R stands for remainder), and we want to be able to estimate R_n. An estimate for R_n will give us some idea of how far off from the truth s_n actually is.

Notice that $R_n = a_{n+1} + a_{n+2} + \ldots$ is just the same as the series that gives s, except that we removed the first n terms. So when we say we want to estimate R_n, we are trying to do something very similar to what we were already doing with the integral test, where we used the value of a definite integral to estimate various partial sums.

To estimate R_n from above, we draw the picture of rectangles over intervals $[i-1, i]$ of height a_i, for $i \geq n+1$. These rectangles are contained in the area under the graph of $f(x)$ from $x = n$ to ∞. Then we have the inequality

$$R_n \leq \int_n^\infty f(x)\,dx.$$

Since the original series passed the integral test, the improper integral on the right is going to converge. If you know how to evaluate that integral, then you will have an upper bound (an estimate for R_n which bounds its size from above) for R_n.

On the flip side, if we draw the picture of rectangles with base $[i, i+1]$ with height a_i for $i \geq n+1$, then we see that these rectangles contain the area under the graph of $f(x)$ from $x = n+1$ to ∞, so we have the inequality

$$\int_{n+1}^\infty f(x)\,dx \leq R_n.$$

This gives a lower bound for R_n. There was nothing special about $f(x) = 1/x^2$ in these calculations:

Remainder theorem for the integral test Suppose the series $\sum_{n=1}^{\infty} a_n = s$ passes the integral test with the function $f(x)$. Then the error for the nth partial sum, which is equal to $s - s_n = R_n$, satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx.$$

Example. Going back to $\sum_{n=1}^{\infty} 1/n^2$, we saw that $s_{100} \approx 1.63498$. So $R_{100} = s - s_{100}$. Since

$$\int_n^{\infty} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_n^{\infty} = \frac{1}{n},$$

the remainder theorem for the integral test tell us that

$$\frac{1}{101} \le R_{100} \le \frac{1}{100}.$$

One way of interpreting this is that

$$s_{100} + \frac{1}{101} \le s \le s_{100} + \frac{1}{100},$$

so that taking 100 terms of this series gives us a value is off by about 0.01. Furthermore, it restricts the value of s to an interval of length $1/100 - 1/101 = 1/(101)(100) \approx 0.0001$, so we get almost four decimal digits of accuracy. This isn't very good for the amount of work it takes to add up 100 terms.

Notice that even though we know that s_{100} is off by about 0.01, the fact that we also have a lower bound for R_n actually helps us, since it allows us to restrict s to an interval of length smaller than 0.01 (in this case, about length 0.0001).

In actuality, $s \approx 1.644934$, which matches with what we just found out (we are off by about 0.01, and adding this to s_{100} gives a value which has four decimal digits of accuracy). If you are curious as to what s actually is, $s = \pi^2/6$!