MATH 8 FALL 2010 CLASS 28, 11/22/2010

1. Local maxima and minima

A common question in single variable calculus is determining the local (or absolute) minima and maxima of a function $y = f(x)$. Recall that we solve this problem by defining a critical point to be a point at which $f'(x) = 0$ or does not exist. Then a result of Fermat says that every local extremum must occur at a critical point (although not every critical point is a local extremum). To test what happens at each local extremum, a useful aid is the second derivative test, where the sign of the second derivative tells can give information on whether we have a local maximum or minimum. In the situation where $f''(x) = 0$, the second derivative test is inconclusive.

Our goal is to develop such tools for use in multivariable calculus. We start by making the analogous definition of what a local maximum or minimum is:

Definition. Let $f(x, y)$ be a function of two variables. The point (a, b) is called a local maximum of $f(x, y)$ if, for all (x, y) near (a, b), $f(x, y) \leq f(a, b)$. Similarly, (a, b) is called a local minimum of $f(x, y)$, if, for all $f(x, y)$ near (a, b), $f(x, y) \geq f(a, b)$. The values of $f(a, b)$ are called the local maximum or local minimum values of f, respectively.

Example. Consider $f(x, y) = |x+y|$. Then all points $(x, -x)$ are local minima for $f(x, y)$, since $f(x, y) \geq 0$ always, and $f(x, -x) = |x - x| = 0$. One can show that $f(x, y)$ has no local maxima, since, at any point (x, y), there is always a direction we can go in which will increase $f(x, y)$.

Suppose we are at a local maximum or minimum (a, b) of a function $f(x, y)$, and $f(x, y)$ is differentiable at (a, b). What can we say about the partial derivatives or gradient of $f(x, y)$ at this point? For example, $f_x(a, b)$ depends only on the values of $f(x, b)$ when x is near a. In particular, because $f(x, y)$ has a local extremum at (a, b), $f(x, b)$, when thought of as a function of just the single variable x, must have an extremum at $x = a$. This means that $f_x(a, b) = 0$, since the derivative of $f(x, b)$ at $x = a$ is just $f_x(a, b)$. Similarly, $f_y(a, b) = 0$. Therefore, we have shown that if (a, b) is a local extremum of $f(x, y)$, and f is differentiable at (a, b), then the partial derivatives must equal 0.

Motivated by this finding, we define a critical point or stationary point of $f(x, y)$ to be any point at which the first partial derivatives of f both vanish, or one of the partial derivatives does not exist. These are candidate points for local extrema; however, we still need some method of distinguishing local maxima, local minima, and points which are neither from each other.

Example. Consider $f(x, y) = y^2 - x^2$. What are the critical points of this function? Are those critical points local maxima, local minima, or neither? We begin by calculating the partial derivatives of $f(x, y) : f_x = -2x, f_y = 2y$. Therefore, there is only one critical point of this function: the point $(0, 0)$. However, notice that this is not a local extrema. For example, if we look at points $(0, y)$, then $f(0, y) = y^2 \geq 0$, so there are always points near $(0, 0)$ for which f is positive. On the other hand, $f(x, 0) = -x^2 \leq 0$, so there are also points near $(0, 0)$ for which f is negative. Therefore, $(0, 0)$ is not a local extrema. As a matter of fact, a graph of $z = y^2 - x^2$ reveals that this function has a saddle-shaped graph. We sometimes call a point which is a critical point at which both partial derivatives exist, but is not a local extremum, a saddle point.

Our next goal will be to describe the analogue of the second derivative test, which will allow us to (in most cases) distinguish local extrema from each other.

Recall that we were discussing local maxima and minima for functions of two variables. We saw that if (a, b) is a local maximum or minimum for $f(x, y)$, and f_x, f_y both exist at (a, b), then the values of both partial derivatives must be equal to 0. In analogy with the single-variable case, we call points (a, b) critical points or stationary points if either $f_x(a, b) = f_y(a, b) = 0$, or if either one of the partial derivatives does not exist at (a, b).

2. The second derivative test

We now discuss the second derivative test. Unlike many other topics we have discussed so far (directional derivative, gradient, etc.), the form in which we state the second derivative test does not have an obvious generalization to functions of more than two variables. Such a generalization does exist, but requires a knowledge of linear algebra to understand in any meaningful way.

Theorem. (The second derivative test) Let the second partial derivatives of $f(x, y)$ be continuous near a point (a, b), and let $f_x(a, b) = f_y(a, b) = 0$. Let $D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(ab)^2$. Then

- If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local maximum.
- If $D < 0$, then (a, b) is not a local minimum or maximum.
- If $D = 0$, the second derivative test is inconclusive.

Remarks.

- A convenient way (and the correct way) to remember D is to think of D as the determinant of a 2×2 matrix
$$D = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}.$$
Not only might this make the formula for D easier to remember (notice the pattern in the subscripts of the partial derivatives), this is also the correct way to define D if we want to generalize the second derivative test to functions of more variables.
- One way to help distinguish between the two possible cases when $D > 0$ is to focus on the function $f(x, b)$, which is just a function of x. If (a, b) is a local maximum for $f(x, y)$, then $x = a$ must be a local maximum for $f(x, b)$, but $f''(x, b) = f_{xx}(x, b)$. So we should expect that $f''(x, b) < 0$ at $x = a$ by the second derivative test for single-variable functions.
- When $D = 0$, the second derivative test genuinely is inconclusive – there are examples of functions which satisfy the hypotheses of the second derivative test, with $D = 0$, which have local maxima, local minima, or neither.
- In the situation where $D > 0$, we could just as well test the sign of $f_{yy}(a, b)$ instead of $f_{xx}(a, b)$. Indeed, if $D > 0$, then $f_{xx}(a, b)f_{yy}(a, b) > 0$, since $f_{xy}(a, b)^2 \geq 0$, so $f_{xx}(a, b), f_{yy}(a, b)$ have the same sign.

We will start by using the second derivative test on functions whose local extrema we can easily analyze using other means.

Examples.

- Let $f(x,y) = x^2+y^2$. It's easy to see that this has a local minimum at $(x,y) = (0,0)$, and has no either local extrema. We'll check that the second derivative test confirms this. We have $f_x = 2x, f_y = 2y$, and so the only critical point of $f(x,y)$ is $(0,0)$. Furthermore, $f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$, so $D = 4$. As $D > 0$ and $f_{xx} > 0$, the second derivative test tells us that $(0,0)$ is a local minimum for $f(x,y)$, as expected.
- Let $f(x,y) = 1/(1+x^2+y^2)$. Then $f(x,y)$ has a local maximum at $(x,y) = (0,0)$, since this is the point which minimizes the value of $1+x^2+y^2$. We also have

$$f_x = \frac{-2x}{(1+x^2+y^2)^2}, f_y = \frac{-2y}{(1+x^2+y^2)^2}.$$

Therefore, the only critical point of $f(x,y)$ is equal to $(0,0)$. We now calculate the second partial derivatives:

$$\begin{aligned} f_{xx} = \frac{-2(1+x^2+y^2)^2 + 4x^2(1+x^2+y^2)}{(1+x^2+y^2)^4} &= \frac{-2(1+x^2+y^2)+4x^2}{(1+x^2+y^2)^3} \\ f_{yy} = \frac{-2(1+x^2+y^2)^2 + 4y^2(1+x^2+y^2)}{(1+x^2+y^2)^4} &= \frac{-2(1+x^2+y^2)+4y^2}{(1+x^2+y^2)^3} \\ f_{xy} = \frac{-2x \cdot -2 \cdot 2y}{(1+x^2+y^2)^3} &= \frac{8xy}{(1+x^2+y^2)^3}. \end{aligned}$$

Although these expressions are fairly complicated, they are much simpler at $(x,y) = (0,0)$:

$$f_{xx}(0,0) = -2, f_{yy}(0,0) = -2, f_{xy}(0,0) = 0.$$

Therefore, $D = (-2)(-2) - 0^2 = 4 > 0$, so we have either a local maximum or a local minimum. Since $f_{xx}(0,0) = -2 < 0$, this is a local maximum, as expected.
- Let $f(x,y) = xy$. Then $f_x = y, f_y = x$, so the only critical point of $f(x,y)$ is $(0,0)$. We also have $f_{xx} = 0, f_{yy} = 0, f_{xy} = 1$, so $D = 0 - 1^2 = -1 < 0$. Therefore, $(0,0)$ is not a local maximum or a local minimum; this is also immediately obvious from the expression $f(x,y) = xy$, since $xy > 0$ when (x,y) is in the first quadrant, while $xy < 0$ when (x,y) is in the second quadrant.

Let's now use the second derivative test to analyze the critical points of a function where the local extrema are not immediately obvious.

Example. Find all local minima and maxima of $f(x,y) = 5x^2 + 2y^2 - 2xy + 6x + 6y + 9$. We begin by calculating the first partial derivatives of $f(x,y)$:

$$f_x = 10x - 2y + 6, f_y = 4y - 2x + 6.$$

To find the critical points of $f(x,y)$, we need to simultaneously solve the pair of equations $f_x = 0, f_y = 0$. These equations can be written as $10x - 2y = -6, 4y - 2x = -6$. Therefore, $10x - 2y = 4y - 2x$, or $12x = 6y \Rightarrow 2x = y$. On the other hand, $10x - 2y = -6 \Rightarrow 6x = -6 \Rightarrow x = -1$. Therefore, $(-1,-2)$ is the only critical point of $f(x,y)$. The second derivatives of f are given by

$$f_{xx} = 10, f_{yy} = 4, f_{xy} = -2.$$

Therefore, $D = 10 \cdot 4 - (-2)^2 = 36 > 0$. Since $f_{xx} = 10 > 0$, this means $(-1,-2)$ is a local minimum of $f(x,y)$.

3. Finding absolute minima and maxima

Another notion from single variable calculus is that of absolute maxima or minima. We say that c is an absolute maximum for $f(x, y)$ on the set D if $f(c) \geq f(x)$ for all x in D. Similarly, we say that (a, b) is an absolute maximum for the set D, now a subset of $\mathbb{R}^2$, if $f(a, b) \geq f(x, y)$ for all (x, y) in D. Recall that for certain sets D, it is not necessarily true that an absolute maximum is a local maximum – for example, consider $y = x$, and D any interval $[a, b]$. Then the absolute maximum of $y = x$ occurs at the right endpoint of the interval, which however is not a local maximum for $y = x$.

However, given a single variable function $f(x)$, the only places where absolute maxima or minima can occur on an interval are either at local minima or maxima, or at the endpoints. Therefore, checking for absolute maxima or minima of a function $f(x)$ on a closed interval is only slightly more work than finding local minima or maxima.

A similar phenomenon occurs for functions of two variables. However, because the domain of a two variable function is now some subset of a plane, instead of a line, we no longer have closed intervals. Instead, we might have a closed set in $\mathbb{R}^2$ on which we want to find the absolute maximum or minimum of a function. Intuitively speaking, a closed set is a set which contains its boundary; for example, the set of points satisfying $x^2 + y^2 < 1$ is not closed, because it does not contain its boundary $x^2 + y^2 = 1$, while the set of points $x^2 + y^2 \leq 1$ is closed.

Suppose D is a closed set in $\mathbb{R}^2$. Then it is a fact that any continuous function on D has an absolute maximum and minimum on D. (This is analogous to the extreme value theorem in single variable calculus, where a continuous function $f(x)$ on a closed interval achieves a maximum and minimum value on that interval.) How do we go about finding these absolute extrema?

Just as in the single variable case, we start by finding all local minima and maxima on the interior of D. Once we've done that, we now need to determine the largest and smallest values of $f(x, y)$ on the boundary of D. How we proceed depends on the exact shape of D, but will often reduce to maximizing or minimizing several different single-variable functions.

Example. Find the local extrema of $f(x, y) = x^3 - 2xy + y^2$. Find the absolute maximum and minimum of this function on the set D given by $-1 \leq x \leq 1, -1 \leq x \leq 1$.

We start by calculating the local extrema. Taking partial derivatives, we have

$$f_x = 3x^2 - 2y, f_y = -2x + 2y.$$

These are both equal to 0 exactly when $-2x + 2y = 0 \Rightarrow x = y, 3x^2 - 2y = 0 \Rightarrow 3x^2 - 2x = 0 \Rightarrow x = 0, 2/3$. Therefore, $(0, 0), (2/3, 2/3)$ are critical points for $f(x, y)$. We use the second derivative test to determine if they are local maxima or minima, or neither:

$$f_{xx} = 6x, f_{yy} = 2, f_{xy} = -2.$$

Therefore, $D(0, 0) = 0(2) - (-2)^2 = -4 < 0$, so $(0, 0)$ is not a local extrema, while $D(2/3, 2/3) = 4(2) - (-2)^2 = 4 > 0$, and $f_{xx}(2/3, 2/3) = 4 > 0$, so $(2/3, 2/3)$ is a local minimum. In particular, $f(2/3, 2/3) = 8/27 - 8/9 + 4/9 = -4/27$.

To determine the absolute maximum and minimum of $f(x, y)$ on D, we need to determine the maximum and minimum value of $f(x, y)$ on the boundary of D. This boundary consists of four line segments, given by $x = -1, x = 1, y = -1$, and $y = 1$. We need to manually check how $f(x, y)$ behaves on each line segment.

For example, when $x = -1$, $f(-1, y) = -1 + 2y + y^2 = (y+1)^2 - 2$. From this factorization we can quickly see that $f(-1, y)$ is smallest when $y = -1$, so $f(-1, -1) = -2$. In particular this is smaller than $f(2/3, 2/3)$, so $(2/3, 2/3)$ will not be an absolute minimum on D. Also,

$f(-1, y)$ is largest when $y = 1$, where $f(-1, 1) = 2$. We keep these largest and smallest values on hand to compare them with the largest and smallest value on other line segments.

When $x = 1, f(1, y) = 1 - 2y + y^2 = (y - 1)^2$. This has smallest value at $y = 1$, where $f(1, 0) = 0$, and largest value when $y = -1$, where $f(1, -1) = 4$.

When $y = 1, f(x, 1) = x^3 - 2x + 1$. To determine the smallest and largest values of this function on $[-1, 1]$, we need to calculate $f'(x, 1) = 3x^2 - 2$. This is equal to 0 when $x = \pm\sqrt{2/3}$. At the endpoints $x = -1, 1$, we have $f(-1, 1) = 2, f(1, 1) = 0$. At $x = \sqrt{2/3}$,

$$x^3 - 2x + 1 = \frac{2\sqrt{2}}{3\sqrt{3}} - \frac{2\sqrt{2}}{\sqrt{3}} + 1 = \frac{-4\sqrt{2}}{3\sqrt{3}} + 1.$$

Even though this number is hard to approximate precisely, we can roughly estimate $4\sqrt{2} \approx 5.6, 3\sqrt{3} \approx 5.2$ so the value of $f(x, 1)$ at $x = \sqrt{2/3}$ is about $-1.1 + 1$. (A more precise approximation is $-1.0887 + 1$.) The value of $f(x, 1)$ at $x = -\sqrt{2/3}$ is then about 2.1, which is larger than $f(-1, 1) = 2$, but smaller than $f(1, -1) = 4$.

Finally, when $y = -1, f(x, -1) = x^3 + 2x + 1$. $f'(x, -1) = 3x^2 + 2 > 0$, so there are no local minima or maxima on this interval. At the endpoints, $f(-1, -1) = -2, f(1, -1) = 4$.

If we look at the maximum and minimum values we have computed for every part of the boundary, we see that the absolute maximum occurs at $(1, -1)$, where $f(1, -1) = 4$, while the absolute minimum occurs at $(-1, -1)$, where $f(-1, -1) = -2$.