MATH 8 FALL 2010 CLASS 26, 11/17/2010

1. Tangent planes

Recall that the derivative of a single variable function can be interpreted as the slope of the tangent line to the graph of the function. We seek an analogous interpretation of the partial derivatives for multivariable functions.

The first and immediate difficulty which presents itself is the fact that there is no distinguished tangent line to a surface. For example, we can cut slices of the surface with various planes (such as when we hold x or y constant, when we take partial derivatives), and then find tangent lines to each curve which appears in a slice. However, no one of these lines by itself could stand in as a linear approximation to the surface, as a tangent line does for a curve.

As a matter of fact, if we are looking for a linear approximation to a surface, we should be looking for a 'tangent plane' instead of a tangent line. Since the graph of a function of two variables is two-dimensional, the linear approximation should also be two dimensional. Let us now think about how we would find a tangent plane.

This tangent plane should evidently contain every tangent line which we can obtain via the slicing procedure described above. Although it is not evident right now that the collection of these tangent lines should form a plane, we will prove this fact later. In any case, if the tangent plane does contain every tangent line as described above, it certainly should contain the tangent lines corresponding to the partial derivatives f_x, f_y.

Suppose we want to find the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0, z_0); of course, $z_0 = f(x_0, y_0)$. Then the two tangent lines corresponding to f_x, f_y have direction vectors $\langle 1, 0, f_x(x_0, y_0)\rangle, \langle 0, 1, f_y(x_0, y_0)\rangle$ respectively. To describe the tangent plane, we need to know a point on the plane, which we do in this case, as well as a normal vector. We have two vectors which lie on the plane; namely, the direction vectors we have found above, and they are not scalar multiples of each other, so their cross product will be a normal vector. We find that this cross product equals

$$\langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$$

so the tangent plane will have equation

$$-f_x(x_0, y_0)x - f_y(x_0, y_0)y + z = -f_x(x_0, y_0)x_0 - f_y(x_0, y_0)y_0 + z_0.$$

We can rearrange this to the form

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

which formally looks like the point-slope form for the equation of a line.

Examples.

- Find the tangent plane to $f(x, y) = xy + y^2$ at $(1, 2)$. This is the same example where we calculated the tangent lines for the slices of this function obtained by holding x or y constant. We find $f_x = y, f_y = x + 2y$, so $f_x(1, 2) = 2, f_y(1, 2) = 5$. The equation for the tangent line is thus

$$z - 6 = 2(x - 1) + 5(y - 2).$$

One can check, for instance, that the two tangent lines we calculated earlier both line on this plane.

- Find the tangent plane to $f(x,y) = xe^y$ at $(2,0)$. Again, we calculate $f_x = e^y, f_y = xe^y$, so $f_x(2,0) = 1, f_y(2,0) = 2$. Also, $f(2,0) = 2$, so the equation for the tangent line is

$$z - 2 = 1(x-2) + 2(y-0) = x - 2 + 2y.$$

We can use this tangent plane to help us approximate value of $f(x,y)$ near a point (a,b) which we can calculate the tangent plane at, when (x,y) is near (a,b).

Example. Use the tangent plane to estimate $f(x,y) = \sqrt{19 - x^2 - y}$ at the point $(2.9, 1.1)$. To solve this problem, we find a point which is close to $(2.9, 1.1)$ at which we can easily evaluate $f(x,y)$, f_x, and f_y. We begin by calculating the partial derivatives of $f(x,y)$:

$$\begin{aligned} f_x(x,y) &= \frac{1}{2\sqrt{19-x^2-y}} \cdot (-2x) = \frac{-x}{\sqrt{19-x^2-y}} \\ f_y(x,y) &= \frac{1}{2\sqrt{19-x^2-y}} \cdot (-1) = \frac{-1}{\sqrt{19-x^2-y}} \end{aligned}$$

Notice that $(3,1)$ is a point near $(2.9, 1.1)$ at which we can easily evaluate f and its partial derivatives. Indeed, we see that

$$f(3,1) = 3, f_x(3,1) = -1, f_y(3,1) = -1/3.$$

Therefore, an equation for the tangent plane to $f(x,y)$ at $(3,1)$ is given by

$$z - 3 = -(x-3) - 1/3(y-1).$$

This tangent plane approximates $f(x,y)$ near $(3,1)$. To find the approximation for $f(2.9, 1.1)$, we determine the z-coordinate of this plane at the point $(2.9, 1.1)$:

$$z - 3 = -(2.9 - 3) - 1/3(1.1 - 1) = .1 - .1/3 = 1/15 \Rightarrow z = 46/15.$$

This is approximately 3.067 , while the actual value of $f(2.9, 1.1)$ is about 3.096.

An alternate way of expressing the fact that the tangent plane is supposed to be approximating $f(x,y)$ (hopefully well near (a,b)) is as follows: let $L(x,y)$ be the linear function $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(y-b)$. This is the function whose graph gives the tangent plane to $f(x,y)$ at (a,b). Then $L(x,y)$ is called the *linearization* of $f(x,y)$ at (a,b). The analogue of this function for functions of a single-variable is $L(x) = f(a) + f'(a)(x-a)$, which is the function whose graph is the tangent line to $y = f(x)$ at $x = a$.

2. Definition of differentiability

Recall that an alternate characterization of the derivative of a single-variable function at a point $x = a$ is that it is the number $f'(a)$ such that

$$\lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} = 0.$$

This formula tells us that the line passing through $(a, f(a))$ with slope $f'(a)$ is the best linear approximation to f near a. In a similar fashion, we might say that a function $f(x,y)$ is differentiable at (a,b) if we can write

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \varepsilon_1(x-a) + \varepsilon_2(y-b)$$

where $\varepsilon_1, \varepsilon_2$ are both going to 0 as $(x,y) \to (a,b)$. In other words, we want the tangent plane to $f(x,y)$ at (a,b) to be a good linear approximation to $f(x,y)$ near (a,b). An alternate way of writing this is to let $\Delta x = (x-a), \Delta y = (y-b)$, and $\Delta z = f(x,y) - f(a,b)$. Then the above equation becomes

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.$$

This definition might seem confusing, but in practice most functions we see, which have partial derivatives, are also differentiable in this stronger sense. This is thanks to the following theorem:

Theorem. Suppose $f(x,y)$ has continuous partial derivatives f_x, f_y at (a,b), and $f(x,y)$ is defined for all (x,y) near (a,b). Then $f(x,y)$ is differentiable at (a,b).

Even though this theorem might seem obvious, there are examples of functions which have partial derivatives f_x, f_y yet are not differentiable! (See, for instance, page 930 of the textbook.)

3. Differentials

Recall that for a function of a single variable, if $y = f(x)$, we sometimes write

$$\frac{dy}{dx} = f'(x)$$

or, if we think of dy, dx as objects we can algebraically manipulate (in a slight abuse of notation),

$$dy = f'(x)\, dx.$$

For example, we think of manipulations like this when we make u-substitutions. If we think of dx as a small change in x, then the equation $dy = f'(x)\, dx$ simply expresses the fact that the tangent line to $f(x)$ is a good linear approximation to $f(x)$, at least near the point of tangency.

In a similar way, if we have $z = f(x,y)$, we sometimes write

$$dz = f_x(x,y)\, dx + f_y(x,y)\, dy$$

to express the fact that the tangent plane to $f(x,y)$ at a point is a good linear approximation to $f(x,y)$ near the point of tangency. We sometimes call $dz = f_x(x,y)\, dx + f_y(x,y)\, dy$ the <u>total differential</u> of $f(x,y)$. If we think of dx, dy as small changes in x, y, and dz as the corresponding change in z in the tangent plane, then this equation can serve as a quicker way to approximate functions using tangent planes.

What's the difference between dz and Δz? The former is the change in the height of the tangent plane, while the latter is the actual change in the value of the function $f(x,y)$. In analogy with the single-variable case, if say $f(x) = x^2$, and $a = 1, dx = 0.1$, then $dy = f'(1)dx = 2 \cdot 0.1 = 0.2$, while $\Delta y = (1.1)^2 - (1.0)^2 = .21$. In practice, you should think of dz as being perhaps an approximation to Δz, which is easier to calculate.

Examples.

- The area of an ellipse with axes of length $2a, 2b$ is given by the formula $A = ab\pi$. (We showed this in a previous class.) Suppose we have an ellipse with axes of length $4, 6$. If we increase the axis of length 4 by .1 and the axis of length 6 by .2, use linearization to approximate the increase in the area of the ellipse.

  We have $A = ab\pi$, so $dA = b\pi\, da + a\pi\, db$. In our problem, $a = 2, b = 3$ (or vice versa), and $da = 0.05, db = 0.1$. Therefore, $dA = 3\pi(.05) + 2\pi(.1) = .35\pi$. Thus the area increases by approximately $.35\pi$.

  Of course, in this example, we could have just calculated the area using $A = ab\pi$ at $a = 2.05, b = 3.1$, and taken a difference.
- Of course, linearization and the differential notation also extends to functions of more than two variables. For example, suppose we have a rectangular solid with sides of length l, w, h. Then the volume of the prism is given by $V = lwh$, and the total differential of V is

  $$dV = wh\, dl + lh\, dw + lw\, dh.$$

  For example, if we have a rectangular prism with sides of length $l = 2, w = 4, h = 5$, and then the sides increase by $.1, .2, .05$ respectively, then an approximation to the change of the volume is given by

  $$4 \cdot 5 \cdot .1 + 2 \cdot 5 \cdot .2 + 2 \cdot 4 \cdot .05 = 2 + 2 + .4 = 4.4.$$