MATH 8 CLASS 23 NOTES, 11/12/2010

We now begin to seriously study multivariable calculus. While vector-valued functions were functions $f : \mathbb{R} \to \mathbb{R}^n$ (or more generally, functions with domain some subset of real numbers), we now want to consider functions which take several variables as input and then output a real number. These functions, which we often also write f, have some subset of numbers in $\mathbb{R}^n$ as domain (for fixed n), and output real numbers.

In general, we will write $f(x_1, \ldots, x_n)$ for such a function, where the x_i are the various variables, but in the case where we look at functions of two variables, we will often write $f(x, y)$ instead. Our goal is to build up a theory of calculus for these functions which generalizes calculus for single-variable functions, but before we can do that we spend some time thinking about how to geometrically visualize and graph functions of two variables.

When drawing a graph of a function $f(x, y)$, we sketch the set of points (x, y, z) such that $z = f(x, y)$. This is in exact analogy to when we draw a graph of a single-variable function $f(x)$, where we sketch the set of points (x, y) such that $y = f(x)$. Sketching graphs of single-variable functions without a calculator or computer is already fairly hard, so one can imagine that sketching graphs of two variable functions would be substantially more difficult, and that is indeed true.

The first task when we sketch the graph of a function $f(x, y)$ is to first determine the domain of f. We typically let the domain be the largest set of numbers in $\mathbb{R}^2$ such that f is defined.

Examples.

- The domain of $f(x, y) = x^2y + xy^3$ is all of $\mathbb{R}^2$. This is an example of a multivariable polynomial, which is a function that can be written as a sum of terms x^ay^b, where a, b are non-negative integers.
- The domain of $f(x, y) = \ln xy$ is the first and third quadrants, not including their boundaries. Indeed, ln is only defined when its argument is positive, so $f(x, y)$ is only defined when $xy > 0$. This is exactly the set of points in the first and third quadrants, without their boundaries.
- The domain of $f(x, y) = \sqrt{4 - x^2 - y^2}$ is the set of points (x, y) with $x^2 + y^2 \leq 4$; this is a disc with radius 2 centered at the origin.
- The domain of $f(x, y) = 1/xy$ is $\mathbb{R}^2$ with the x and y axes removed.

In some situations it may be possible to easily determine the range of $f(x, y)$, which is the set of numbers z such that $z = f(x, y)$ for some (x, y). We use the four examples above again:

Examples.

- $f(x, y) = x^2y + xy^3$ has range all real numbers. Indeed, notice that $f(1, y) = y + y^3$, which as a function of y has range all real numbers.
- $f(x, y) = \ln xy$ has range equal to all real numbers as well; for example, $f(x, 1) = \ln x$ has range all real numbers.
- The range of $f(x, y) = \sqrt{4 - x^2 - y^2}$ is $[0, 2]$, since the expression under the square root takes every value in $[0, 4]$ and no other positive values.
- The range of $f(x, y) = 1/xy$ is all nonzero real numbers.

When we want to sketch a graph of $z = f(x, y)$, we will often find ourselves having lots of difficulty, since it it is much harder to visualize what is happening with a two-variable function than a single-variable function. We begin by examining a few simple, model examples.

Example. Graph $z = -x - y + 1$. Notice that this is a plane; its intersections with the x, y, z axes are $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ respectively, and these three points completely determine the plane. We can certainly sketch the triangle these three points determine and then visualize the plane extending off in all directions.

Two very handy ideas in sketching graphs of two-variable functions are level curves and contours. A level curve of $f(x, y)$ is a graph, in the plane where (x, y) are defined, of the set of points (x, y) which satisfy $f(x, y) = C$ for some constant C. We often draw several such level curves, for various C which are equally spaced, to get an idea of how $f(x, y)$ changes as x, y change.

The idea of level curves is probably not unfamiliar to you. For example, a topographical map (such as one you might use when hiking or mountain climbing) is a map which contains curves that indicate all the points on the map which are of equal altitude. In this situation, we can think of the level curves as being the curves $f(x, y) = C$ where x, y are the coordinates (such as latitude and longitude) of a point on Earth, while C is the height of the level curve in question.

If we graph several level curves with C equally spaced, then the spacing of the level curves can give us an intuitive idea of how rapidly $f(x, y)$ is changing. Regions where many curves are tightly clustered indicate regions where $f(x, y)$ is rapidly changing, while regions where curves are sparse indicate regions where $f(x, y)$ is not changing much at all.

Example. Sketch the level curves of $f(x, y) = \sqrt{16 - x^2 - y^2}$ where $C = 0, 1, 2, 3, 4$.

We plot $0 = \sqrt{16 - x^2 - y^2} \Rightarrow x^2 + y^2 = 16$. This is a circle with radius 4. Similarly, for $C = 1$, $1 = \sqrt{16 - x^2 - y^2} \Rightarrow x^2 + y^2 = 15$, which is a circle with radius $\sqrt{15}$. We can do the same for $C = 2, 3, 4$ to find circles of radius $\sqrt{12}, \sqrt{7}$, and 0, respectively. Notice that the curves are more tightly bunched when $x^2 + y^2$ is larger; this indicates that this function is changing more rapidly there than it is near the origin.

Incidentally, if we let $z = f(x, y)$, then we see that $x^2 + y^2 + z^2 = 16$, so this graph is actually the top half of a sphere with radius 4, centered at the origin.

Example. Sketch the level curves of $f(x, y) = x^2 + y^2$ for $C = 1, 2, 3, 4, \ldots$. Again, we get concentric circles of radius $\sqrt{n}$, for the level set corresponding to $C = n$. Notice that these level sets get more and more tightly bunched as n increases, which shows that $f(x, y)$ is increasing faster and faster as $x^2 + y^2$ gets large. The graph of this function is called an elliptic paraboloid.

The contours of a graph $z = f(x, y)$ are curves drawn on the graph which indicate all points with the same z-coordinate. In other words, contours are just the lifts of level curves from the xy-plane to the actual graph of the function $f(x, y)$, and level curves are just projections of contours onto the xy-plane.

Example. Sketch the level curves of $f(x, y) = \sqrt{x^2 + y^2}$ for $C = 1, 2, 3, \ldots$. This time, the level curve corresponding to $n = C$ is $x^2 + y^2 = n^2$, which is a circle of radius n. These level curves are evenly spaced concentric circles, which implies that $f(x, y)$ grows in a uniform rate. As a matter of fact, $z = f(x, y)$ is a cone. One way to see this is that if we consider

$y = 0$, then $z = |x|$, which is the cross-section for a cone. This is one way to distinguish between an elliptic paraboloid ($z = x^2 + y^2$) and a cone ($z = \sqrt{x^2 + y^2}$).

Example. Sketch the level curves of $f(x, y) = \sin(x - y)$, for $C = -1, -1/2, 0, 1/2, 0$. This time, the five values of C correspond to $x - y = -\pi/2, -\pi/6, 0, \pi/6, \pi/2$, and any translate by an integer multiple of 2π. These are lines of slope 1, with different y-intercepts. Notice that these lines are not evenly spaced, which reflects the fact that sin does not have constant rate of change.

Example. Sketch the level curves of $f(x, y) = 1/xy$, for $C = -2, -1, 1, 2$. This time, we obtain various hyperbolas. This function is undefined on the x and y axes. What might the graph of this function look like?

A remark on calculator and computer graphing

In this class, we frequently try to tell students to not use calculators for calculations so as to build mental intuition. When it comes to graphing in three-dimensions, however, calculators and computers can be very useful assistants. By no means should you rely on them – in particular, you are expected to know how to sketch level curves and interpret sketches of level curves (such as identifying when a function is increasing rapidly or not), but computer software can plot graphs that would be very difficult to draw by hand. They also have the advantage of being easily changeable (you can vary parameters without much effort to see how graphs would change) as well as rotatable! When looking at three-dimensional objects, the ability to rotate a picture makes a huge difference.

Most graphing calculators (such as the popular TI calculators) have the capability to at least draw primitive three-dimensional plots. However, for fuller features you should consider obtaining a computer package. There are fairly simple three-dimensional graphing calculator programs for computers, but a software program like Mathematica or Maple might be worth getting. It isn't very hard to plot in several dimensions using these programs, and they also have many, many other features which might be handy in other classes. Dartmouth does not have a license for Mathematica, but you can obtain a fairly old version (Maple 11, vs the current Maple 13) of Maple from the Dartmouth computing website.

You can go to http://www.dartmouth.edu/comp/resources/downloads/ and then pick the appropriate operating system to find out more about obtaining Maple. You are not required to use it in this class, but it will probably prove to be very helpful. Do not rely exclusively on it, though – as mentioned before, you should use it as a supplement to enhance your intuition, not as a tool which you use without much thought!