MATH 8 CLASS 21 NOTES, 11/08/2010

1. Arc length of curves

Now that we know how to take derivatives of vector-valued functions, we briefly describe how to solve a natural and common geometric problem. Suppose we have a curve, perhaps given by a vector-valued function $\vec{r}$. If $\vec{r}$ is differentiable on an interval $[t_1, t_2]$, then it draws out some continuous curve from $\vec{r}(t_1)$ to $\vec{r}(t_2)$. A natural question to ask is how long this curve is.

Recall that a definite integral $\int_a^b f(x)\,dx$ can be interpreted as the signed area under the graph of $f(x)$ from a to b, and is calculated by using rectangles of smaller and smaller width to approximate this area. In a similar way, given a curve, we might approximate its length by using shorter and shorter line segments, which give better and better approximations the smaller they get. For example, we might use line segment connecting $\vec{r}(t)$ and $\vec{r}(t+h)$ as an approximation for the length of the curve from $\vec{r}(t)$ to $\vec{r}(t+h)$.

The length of this line segment can be easily determined in terms of the component functions of $\vec{r}$. For example, if $\vec{r}(t) = \langle x(t), y(t)\rangle$, then the length of the segment, which we call ℓ, connecting $\vec{r}(t) = (x(t), y(t))$ and $\vec{r}(t+h) = (x(t+h), y(t+h))$ is given by the Pythagorean Theorem:

$$\ell^2 = (x(t+h) - x(t))^2 + (y(t+h) - y(t))^2$$

Notice that $(x(t+h) - x(t))^2$ can be approximated by $x'(t)h$, and this approximation becomes better and better as $h \to 0$. Therefore, it is plausible that the arc length of the curve traced out by $\vec{r}$ from t_1 to t_2 is given by the definite integral

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_{t_1}^{t_2} \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

Let us begin by checking that this formula for arc length matches the lengths of curves that we already know the lengths of, from geometry:

Examples.

- Consider the line segment from (x_1, y_1) to (x_2, y_2). This line segment has length

  $$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

  by the Pythagorean Theorem. We can also parametrize this line segment (in a one-to-one fashion) using the function $\vec{r}(t) = \langle x(t), y(t)\rangle, 0 \le t \le 1$, where $x(t) = x_1 + (x_2 - x_1)t, y(t) = y_1 + (y_2 - y_1)t$. Then $x'(t) = (x_2 - x_1), y'(t) = (y_2 - y_1)$, so the arc length formula gives

  $$\int_0^1 \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,dt = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

  as expected.
- Calculate the circumference of the unit circle using the arc length formula. We can parameterize the unit circle by letting $\vec{r}(t) = \langle \cos t, \sin t\rangle$ where $0 \le t \le 2\pi$. Then $x'(t) = -\sin t, y'(t) = \cos t$, and the arc length formula reads

$$\int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} dt = 2\pi$$

again as expected. Notice that if we had let t range from 0 to 4π, say, then $\vec{r}(t)$ traces out the unit circle twice. Even though the circle still has circumference 2π, the arc length formula gives a value of 4π, because the circle had been traced out twice. It is perhaps more accurate to think of the arc length formula as giving the distance a particle travels if its position is given by $\vec{r}(t)$; nevertheless, as long as this particle traverses a curve in such a way so that it visits each point of that curve exactly once, the arc length formula will give the length of the curve.

There is nothing special about two dimensions. If $\vec{r}(t) = \langle r_1(t), \ldots, r_n(t)$, then the arc length of the curve from $\vec{r}(t_1)$ to $\vec{r}(t_2)$ is

$$\int_{t_1}^{t_2} \sqrt{r_1'(t)^2 + \ldots + r_n'(t)^2}\, dt.$$

Example. Calculate the arc length of the curve given by $\vec{r}(t) = \langle t, t^2, 2t^3/3\rangle$ for $0 \leq t \leq 2$. We see that $\vec{r}\,'(t) = \langle 1, 2t, 2t^2$, so the arc length of this curve is given by

$$\int_0^2 \sqrt{1^2 + (2t)^2 + (2t^2)^2}\, dt = \int_0^2 \sqrt{1 + 4t^2 + 4t^4}\, dt.$$

Notice that we are in the fortunate position where we can factor the term inside the square root. We obtain

$$\int_0^2 \sqrt{(1+2t^2)^2}\, dt = \int_0^2 1 + 2t^2\, dt = t + \frac{2t^3}{3}\Big|_0^2 = 2 + 16/3 = 22/3.$$

Finally, let us consider the special case of arc lengths of graphs of functions $y = f(x)$ from x_1 to x_2. Any graph of a function can be parameterized by $x = t, y = f(t)$. Therefore, $x'(t) = 1, y'(t) = f'(t)$, so in this case, the arc length for the graph of $f(x)$, from x_1 to x_2, is given by

$$\int_{x_1}^{x_2} \sqrt{1 + f'(x)^2}\, dx.$$

Example. Calculate the length of the parabola $y = x^2$ from $x = 0$ to $x = 2$. Since $y' = 2x$, we want to calculate the integral

$$\int_0^2 \sqrt{1 + 4x^2}\, dx.$$

This has the form of an integral we should use a trigonometric substitution on. Recalling our work from two weeks ago, we use the substitution $x = \tan\theta/2$. Then $dx = \sec^2\theta/2\, d\theta$, and the bounds $x = 0, 2$ become $\theta = 0, \arctan 4$. Then the integral becomes

$$\int_0^{\arctan 4} \sqrt{1 + \tan^2\theta}\, \frac{\sec^2\theta}{2}\, d\theta = \int_0^{\arctan 4} \sec^3\theta\, d\theta.$$

Recall that to evaluate this integral, we need to use a 'reduction formula' obtained from integration by parts, which was on a homework assignment from several weeks ago. Applying this reduction formula gives

$$\frac{1}{2}\int \sec^3\theta\,d\theta = \frac{1}{2}\left(\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\,d\theta\right).$$

This last integral we know how to evaluate. Therefore, the definite integral we are interested in is equal to

$$\frac{1}{4}\left(\tan\theta\sec\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\arctan 4}.$$

At $\theta = 0$, the above expression is also equal to 0, because $\tan\theta = 0, \sec\theta = 1$. At $\theta = \arctan 4$, we have $\tan\theta = 4$. To calculate $\sec\theta$, draw a right angle triangle with angle θ satisfying $\tan\theta = 4$. For example, a right angle triangle with opposite side 4, adjacent side 1, and hypotenuse $\sqrt{17}$ works. Therefore, $\sec\theta = \sqrt{17}$. Then the above expression at $\theta = \arctan 4$ is equal to

$$\frac{1}{4}(4\sqrt{17} + \ln(4+\sqrt{17})) = \sqrt{17} + \frac{1}{4}\ln(4+\sqrt{17}).$$

It is remarkable that to calculate the arc length of a curve as simple as $y = x^2$, we obtain trigonometric integrals and logarithms!

2. Velocity and Acceleration

One of the most useful and immediate applications of differentiation and integration is the ability to calculate the velocity and acceleration of an object given its position. When we first learn this in Calculus I, we are restricted to the case where the position of the object is given by a single variable function. In real life, however, objects move in two or three dimensions. Now that we have the mathematical terminology to easily describe motion in multiple dimensions, it is not difficult to generalize the ideas of velocity and acceleration.

Suppose a particle moves in such a way so that its position at time t is given by the vector-valued function $\vec{r}(t)$. We will often assume that $\vec{r}$ takes values in $\mathbb{R}^2$ or $\mathbb{R}^3$ to be realistic, although in principle we could work in $\mathbb{R}^n$. The velocity of the particle is then defined to be the derivative $\vec{v}(t) = \vec{r}\,'(t)$. Similarly, the acceleration of the particle is defined to be $\vec{a}(t) = \vec{r}\,''(t)$. Of special note is the speed of the particle, which is equal to $|\vec{v}(t)|$, which is the norm of the vector $v(t)$.

Examples.

- Suppose a particle moves in the plane with position function $\vec{r}(t) = \langle a\cos t, b\sin t\rangle$ where a, b are positive real numbers. What is the velocity and acceleration of this particle? What shape is the path of the particle? The velocity is $\vec{v}(t) = \langle -a\sin t, b\cos t$, while the acceleration is $\langle -a\cos t, -b\sin t$. The shape is an ellipse with axes of length $2a, 2b$.
- Suppose a particle has acceleration given by $\vec{a}(t) = \langle 1, e^t, \sin t$, and has initial velocity and acceleration $\vec{v}(0) = \langle 1, 0, 2\rangle, \vec{r}(0) = \langle 1, 1, 1\rangle$. What is $\vec{r}(t)$?

  Like any sort of problem of a single-variable, we integrate twice, using the initial condition to determine what the constants of integration are at each step. For example, integrating $\vec{a}(t)$ component-by-component gives $\vec{v}(t) = \langle t + C_1, e^t + C_2, -\cos t + C_3\rangle$. Since $\vec{v}(0) = \langle 1, 0, 2\rangle$, this means $C_1 = 1, C_2 = -1, C_3 = 3$, so $\vec{v}(t) = \langle t+1, e^t - 1, -\cos t + 3\rangle$. Integrate one more time to obtain

  $$\vec{r}(t) = \left\langle \frac{t^2}{2} + t + C_1, e^t - t + C_2, -\sin t + 3t + C_3 \right\rangle.$$

  Using the initial condition $\vec{r}(0) = \langle 1, 1, 1\rangle$ gives the answer

$$\vec{r}(t) = \left\langle \frac{t^2}{2} + t + 1, e^t - t, -\sin t + 3t + 1 \right\rangle.$$

- A particle's position vector is given by $\vec{r}(t) = \langle \cos t, \sin t, t^2 \rangle$. What is its speed at time t? At what time is the particle moving slowest? The velocity vector for this particle is $\vec{v}(t) = \langle -\sin t, \cos t, 2t \rangle$. Therefore, the speed of the particle is $|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 4t^2} = \sqrt{1 + 4t^2}$. To determine when the speed is a minimum, we probably want to take a derivative of the above function and set that equal to 0. However, taking derivatives of functions inside square roots can be rather messy, so we make the observation that $|\vec{v}(t)|$ is minimized at the same time that $|\vec{v}(t)|^2 = 1 + 4t^2$ is minimized, which is easier to differentiate. We find that

$$\frac{d}{dt}|\vec{v}(t)|^2 = 8t$$

so that $t = 0$ is a critical point. As a matter of fact it is obvious that $|\vec{v}(t)|$ is minimal at $t = 0$, with $|\vec{v}(0)| = 1$, since $1 + 4t^2$ is a parabola with minimum at $t = 0$.

- Suppose a particle moves in such a way so that its distance from the origin is constant. (That is, the particle travels on a sphere with center at the origin.) Show that its velocity is always orthogonal to its position. Let $\vec{r}(t)$ describe the motion of an object: then $\vec{r}(t)$ has constant length. In particular, this means that $\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2$ is constant, so its derivative is equal to 0. On the other hand,

$$(\vec{r}(t) \cdot \vec{r}(t))' = 2\vec{r} \cdot \vec{r}\,' = 0,$$

which implies that $\vec{r} \cdot \vec{r}\,' = 0$, or that $\vec{r}, \vec{r}\,'$ are orthogonal, as desired.