MATH 8 CLASS 19 NOTES, 11/03/2010

Last time, we saw how to write equations for lines that lie in $\mathbb{R}^3$. In particular, a line is completely specified by a point on that line and a direction vector; for example, given a point $P = (x_0, y_0, z_0)$ and direction vector $\vec{v} = \langle a, b, c\rangle$, we have the line ℓ given parametrically by $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$, where t is any real number. There is also a way to convert the parametric form of a line to a symmetric form, and vice versa.

Lines are one-dimensional objects in $\mathbb{R}^3$. We now seek to understand corresponding linear two-dimensional objects in $\mathbb{R}^3$: that is, we seek to understand planes. A basic geometric fact is that given any three non-collinear points in $\mathbb{R}^3$, there exists a unique plane which passes through them. We now seek to understand how we can algebraically specify the points which constitute a plane.

1. The equation of a plane

Notice that given any plane, there exists a nonzero vector which is orthogonal to any vector which 'lies' in that plane: that is, given any two points P, Q on a plane, there is a nonzero vector which is orthogonal to every vector $\vec{PQ}$, regardless of the choice of P, Q.

For example, given the xy-plane, the vector $\langle 0, 0, 1\rangle = \vec{k}$ is orthogonal to every vector that lies in the xy-plane, because all such vectors have z-coordinate equal to 0. Such a vector, which must be nonzero, is called a <u>normal vector</u> to the plane. Notice that normal vectors are not unique, because any nonzero scalar multiple of a normal vector still is a normal vector. Nevertheless, this is the extent of the non-uniqueness, as it is possible to show that any two normal vectors to a given plane are scalar multiples of each other.

Suppose we have a plane V, and we know that $\vec{n} = \langle a, b, c\rangle$ is a normal vector for V. How can we find an equation which determines V? Let (x_0, y_0, z_0) be an arbitrary but fixed point on V, and let (x, y, z) be a general point on V. Then the vector $\langle x - x_0, y - y_0, z - z_0\rangle$ is orthogonal to $\vec{n}$: that is, their dot product is equal to 0. Then this yields

$$a(x - x_0) + b(y - y_0 + c(z - z_0) = 0 \Rightarrow ax + by + cz = d, \text{where } d = ax_0 + by_0 + cz_0.$$

The equation $ax + by + cz = d$ is called the <u>scalar equation</u> or <u>implicit equation</u> for the plane V. Conversely, given an equation $ax + by + cz = d$, we know that this defines a plane with normal vector $\vec{n} = \langle a, b, c\rangle$. In summary, we can easily find the equation for a plane given its normal vector and one point on the plane.

Suppose, on the other hand, that we are told other information about the plane; for example, perhaps we know the coordinates of three points on the plane. How do we find a normal vector to that plane? The answer is in the dot product! For example, if P, Q, R are non-collinear on the plane, then the vectors $\vec{PQ}, \vec{PR}$ certainly lie on the plane, so their cross product, which is nonzero since P, Q, R are not all on the same line, will be normal to both $\vec{PQ}, \vec{PR}$. It is not too hard to see that $\vec{PQ} \times \vec{PR}$ will be orthogonal to every vector on the plane.

Examples.

- Find the implicit equation for the plane passing through $(0, 0, 0), (1, 3, 2)$, and $(-2, 1, 4)$. In this case, we find that $\vec{v_1} = \langle 1, 3, 2\rangle, \vec{v_2} = \langle -2, 1, 4\rangle$ are both vectors which lie on the plane. Therefore, their cross product will be a normal vector for the plane:

$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 2 \\ -2 & 1 & 4 \end{vmatrix} = \langle 10, -8, 7 \rangle.$$

This vector is a normal vector for the plane. To determine d, we just plug in any point into the formula $d = ax_0 + by_0 + cz_0$. In this case, we use the point $(0, 0, 0)$, and find $d = 0$. Therefore, the implicit formula for this plane is given by $10x - 8y + 7z = 0$. We can check that this answer is correct by plugging in the three original points we were given and verifying that they satisfy the equation for the plane. Also, notice that had we used either of the two points $(1, 3, 2), (-2, 1, 4)$ to determine d, we would have found $d = 0$ as well.

- Find the implicit equation for the plane passing through $(1, 2, -2), (0, 2, 4)$, and $(-1, 3, 3)$. This time, we need to find two non-collinear vectors on this plane. For example, $\vec{v_1} = \langle 1 - 0, 2 - 2, -2 - 4 \rangle = \langle 1, 0, -6 \rangle$ and $\vec{v_2} = \langle 1 - (-1), 2 - 3, -2 - 3 \rangle = \langle 2, -1, -5 \rangle$ work. Then we take their cross product:

$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -6 \\ 2 & -1 & -5 \end{vmatrix} = \langle -6, -7, -1 \rangle.$$

Therefore, the plane has equation $-6x - 7y - z = d$, where $d = -6(0) - 7(2) - 4 = -18$, say. So we find that an implicit equation for this plane is given by $-6x - 7y - z = -18$. If we want to, we can multiply this equation by any nonzero scalar; for example, we can remove the negative sign by multiplying by -1 to get the equation $6x + 7y + z = 18$. It is clear that these two equations describe the same set of points in $\mathbb{R}^3$.

2. Various problems with planes

There are a variety of geometric problems we can ask which involve planes in $\mathbb{R}^3$. For example, we have already solved the problem of finding an equation for a plane given information like three non-collinear points on the plane, or given a point on the plane and a normal vector to the plane. We will briefly look at examples of other types of problems we might encounter. Pay close attention to how the geometry and algebra interact in each of these problems.

2.1. **Intersection of planes with planes or lines.** Suppose we have two planes V_1, V_2. A natural question to ask is what their intersection looks like. Of course, if V_1, V_2 are equal, then their intersection is just $V_1 = V_2$. So suppose we have two distinct planes. Then a bit of thought will suggest that they must either be parallel (have no common intersection), or intersect in a line. How can we distinguish between the two cases, and if two planes intersect, how can we determine an equation for the line of intersection?

Notice that if two planes are identical or are parallel, their normal vectors must be nonzero scalar multiples of each other. Therefore, we can detect whether two planes are parallel or not by examining their normal vectors, which are easy to find if we are given an equation which defines the two planes. If two planes have normal vectors which are not scalar multiples of each other, then the two planes must intersect in a line. If two planes have the same normal vectors (up to scalar multiplication), they are either identical or parallel, and it is easy to distinguish the two cases, since two planes are parallel if their defining equations are scalar multiples of each other.

Examples.

- What is the intersection of $2x + 3y - 4z = 7, 6x + 9y - 12z = 21$? Two normal vectors for these planes are given by $\langle 2, 3, -4\rangle, \langle 6, 9, -12\rangle$ respectively. Notice that these are scalar multiples of each other, so these two planes are either identical or parallel. They are actually identical, since the second equation is just the first equation times 3.
- What is the intersection of $x + 2y - 3z = -2, -2x - 4y + 6z = 9$? Again, these two planes have normal vectors $\langle 1, 2, -3\rangle, \langle -2, -4, 6\rangle$, respectively. These are scalar multiples of each other. On the other hand, these two planes are not identical, since the second equation is not equal to -2 times the first. Therefore, these two planes are parallel.
- What is the intersection of $x - 2y - 3z = 4, -x + 3y + 2z = -2$? The normal vectors for these two planes are $\langle 1, -2, -3\rangle, \langle -1, 3, 2\rangle$, which are not scalar multiples of each other, so they intersect in a line. How can we find an equation which defines this line? Suppose we first want to determine a direction vector for this line. Since this direction vector lies on both planes, it will be orthogonal to both normal vectors. Therefore, we can find a direction vector by taking the cross product of the two normal vectors for our planes. In this case, we have

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -3 \\ -1 & 3 & 2 \end{vmatrix} = \langle 5, 1, 1\rangle.$$

Therefore, the line of intersection has the parametric equation $x = 5t + x_0, y = t + y_0, z = t + z_0$, where (x_0, y_0, z_0) is any point on both planes. To find such a point, we want to simultaneously solve the equations

$$\begin{aligned} x - 2y - 3z &= 4 \\ -x + 3y + 2z &= -2 \end{aligned}$$

There is a systematic way of solving systems of linear equations, such as this one, which you can learn in a linear algebra class. The objective is to eliminate as many variables as possible from each equation. For example, if we add the first equation to the second (one can check this does not change the solution set), we obtain the system

$$\begin{aligned} x - 2y - 3z &= 4 \\ y - z &= 2 \end{aligned}$$

We have eliminated the x from the second equation. We can now eliminate y from the first equation, by adding two copies of the second equation to the first:

$$\begin{aligned} x + 0y - 5z &= 8 \\ y - z &= 2 \end{aligned}$$

At this point, we can let $z = t$, and find that $x = 8 + 5t, y = 2 + t$. Therefore, the line of intersection is given parametrically by $x = 8+5t, y = 2+t, z = t$. Notice that when $t = 0$, we get a point $(8, 2, 0)$ on both planes, and that this general method of solving linear equations was also able to recover the cross product of the two initial normal vectors.

What about the intersection of a line and a plane? Again, some more thought shows that such an intersection must either be empty (no intersection), a point, or a line. In the case where the intersection is empty or the entire line, the direction vector for that line must also lie on the plane, and hence will be orthogonal to the normal vector for the plane. On the other hand, if the intersection point is a point (ie, the direction vector is not orthogonal to the normal vector for the plane), we can solve for the intersection point by plugging in the parametric equation for the line into the equation for the plane.

Examples.

- Find the intersection between $2x + y - 3z = 4$ and $x = t + 3, y = t - 2, z = t$. A direction vector for the line is given by $\langle 1, 1, 1\rangle$. We see that this is orthogonal to the normal vector $\langle 2, 1, -3\rangle$ for the plane, so the intersection is either empty or the entire line. To determine which, we simply plug in the equation for the line into the equation for the plane, and obtain $2(t+3)+(t-2)-3t = 4 \Rightarrow 4 = 4$. This indicates that the entire line lies on the plane. Also, notice that we could have obtained this information by skipping directly to the step where we plugged in the parametric equation for the line into the equation for the plane. Had we considered a line like $x = t + 2, y = t - 2, z = t$, say, then plugging in this parametric equation would have given $2 = 4$, which is always false, so this line does not intersect the plane $2x + y - 3z = 4$ at any point.
- Find the intersection between $-x + y + z = 3$ and $x = 2t - 3, y = t + 1, z = -t + 2$. We plug in the parametric equation for the line into the equation for the plane and obtain $-(2t - 3) + (t + 1) + (-t + 2) = 3 \Rightarrow -2t + 6 = 3 \Rightarrow t = 3/2$. Therefore, the point of intersection is given by $(0, 5/2, 1/2)$. We can check that this answer is correct by verifying that this point lies on both the plane and line in question.

2.2. **The angle between two planes.** Suppose two planes V_1, V_2, with normal vectors n_1, n_2 intersect in a line. Then we call the acute (or possibly right) angle between these two planes the angle between the two planes. Unlike angles between vectors, this angle is always acute or right: that is, between 0 and $\pi/2$ radians. Two intersecting planes form two angles, which are supplementary to each other, so we always choose the angle which is not obtuse.

A bit of thought will show that this angle is the same as the angle formed by the two normal vectors n_1, n_2, except possibly that if the angle between n_1, n_2 is obtuse, then we need to take the supplement of that angle. (This is equivalent to switching one of the normal vectors with its negative and re-calculating the angle.)

Examples.

- Find the angle between $-x + 2y - 3z = 42, 3y + 2z = -10$. These two planes have normal vectors $\vec{n_1} = \langle -1, 2, -3\rangle, \vec{n_2} = \langle 0, 3, 2\rangle$ respectively. Notice that $n_1 \cdot n_2 = 0$; therefore, the angle θ between these two vectors is given by $\cos\theta = 0 \Rightarrow \theta = \pi/2$. This means that the two planes have an angle of $\pi/2$ between them, and hence are orthogonal to each other.
- Find the angle between $2x + 3y + z = 5, x - y - z = 4$. Two normal vectors are $\vec{n_1} = \langle 2, 3, 1\rangle$ and $\vec{n_2} = \langle 1, -1, -1\rangle$. The angle between these two vectors is given by θ in the equation

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|n_1||n_2|} = \frac{-2}{\sqrt{14}\sqrt{3}}.$$

Therefore, $\theta = \arccos -2/\sqrt{42}$. However, this angle is obtuse (since $\cos\theta < 0$), so to find the angle between the two planes we need to take the supplementary angle. This is equivalent to replacing one of n_1, n_2 with its negative, so the angle between these two vectors is $\arccos 2/\sqrt{42}$.

2.3. **The distance of a point from a plane.** We conclude by answering a natural geometric question which arises in a variety of contexts. Suppose we have a point, say $P = (x_1, y_1, z_1)$, and a plane $V : ax + by + cz = d$. What is the distance of P from V? The distance from a point to a plane is defined to be the shortest distance from the point to any point on the plane. In particular, this is achieved when we find a point R on the plane such that $\vec{PR}$ is orthogonal to the plane; that is, parallel to a normal vector for V.

In general, it is difficult to choose R to ensure that this happens. However, suppose we select an arbitrary point Q on V, which is always easy to do. Then although $\vec{PQ}$ is not orthogonal to V, the projection of this vector onto $\vec{n} = \langle a, b, c\rangle$ is orthogonal to V, and its length is the distance of P from V. Therefore, to calculate the distance of a point from a plane, we need only know the coordinates of the point P we are interested in, a normal vector $\vec{n}$ for the plane, and any arbitrary point Q on the plane. The (absolute value of the) scalar projection of $\vec{PQ}$ onto $\vec{n}$ is then the distance of P from V.

Example. Find the distance of $P = (2, 1, -2)$ to $V : x + 2y + 3z = 4$. We begin by noting that $\vec{n} = \langle 1, 2, 3\rangle$ is a normal vector for the plane in question. Also, the point $Q = (0, 2, 0)$ lies on this plane. Therefore, $\vec{PQ} = \langle -2, 1, 2\rangle$. The distance is then given by the absolute value of

$$\frac{\vec{PQ} \cdot \vec{n}}{|\vec{n}|} = \frac{6}{\sqrt{14}}.$$

Therefore, the distance is $6/\sqrt{14}$.