MATH 8 CLASS 18 NOTES, 11/01/2010

1. Lines and vectors in $\mathbb{R}^2$

We now consider the question of how to specify equations for lines in $\mathbb{R}^3$. Let's begin by recalling what we know in $\mathbb{R}^2$. The traditional way to specify a line in $\mathbb{R}^2$ is to use a linear equation $ax + by = c$. However, there is an alternate way to specify a line ℓ.

Start by selecting any point on ℓ, say the point P_0 (which we can either think of as a vector or ordered pair). We then take any nonzero vector whose starting point and end point both lie on ℓ; call this vector $\vec{v}$. Then notice that ℓ can be described as the set of points of the form $P_0 + t\vec{v}$, for any real number t. The vector $\vec{v}$ is sometimes called a *direction vector* for ℓ. We sometimes call this description of ℓ a *parametric* equation for ℓ, because the set of points on this line is described by the parameter t.

Examples.

- Consider the line $y = 2x + 3$. We select any point on this line; say $(0, 3)$. We now need to find a direction vector. For instance, we can look at the vector which starts at $(0, 3)$ and ends at $(1, 5)$. Then a direction vector is given by $\langle 1, 2\rangle$. This means we can describe this line as the set of points satisfying $(x, y) = (0, 3)+t(1, 2) = (t, 3+2t)$.

  Notice that it does not matter what initial point or direction vector we choose; there are infinitely many choices for these, but regardless of the choices we make, we end up with the same line. In this example, any nonzero scalar multiple of $\langle 1, 2\rangle$ would have worked as a direction vector, like $\langle 3, 6\rangle$ or $\langle -2, -4\rangle$. However, be sure that you use a nonzero direction vector.
- Consider the line $x = 2$. Take any point, say $(2, 0)$ on this line. The vector $\langle 0, 1\rangle$ works as a direction vector, so this line can be described by $(2, 0) + t(0, 1) = (2, t)$. Parameterizations like $(2, -2t), (2, 1 + 4t)$, etc. all work as well.

If you think about it, we haven't really done much here, since we basically seem to be replacing x with the parameter t in the equation of a line $ax + by = c$. However, the point of view of a line being specified by a point on the line and a direction vector is probably the easiest way to think about lines in space of dimension greater than 2.

2. Lines in $\mathbb{R}^3$

Recall that a linear equation in $\mathbb{R}^3$, such as $ax + by + cz = 0$, does not define a line, but rather defines a plane. So how do we describe a line in $\mathbb{R}^3$?

Perhaps the simplest answer is the point and direction vector description above. One can still check that a line in $\mathbb{R}^3$ can be specified by giving any point on that line, say a point (x_0, y_0, z_0), and a direction vector $\vec{v} = \langle a, b, c\rangle$. Then that line is given parametrically by the equation $(x_0, y_0, z_0) + t(a, b, c) = (x_0 + at, y_0 + bt, z_0 + ct)$.

There is another less common way to describe a line in $\mathbb{R}^3$. Suppose we know that a line is given parametrically by $(x_0 + at, y_0 + bt, z_0 + ct)$. If $a, b, c \neq 0$, then we can solve for t in the various equations $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$. This gives

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

The last two equations describe a line in $\mathbb{R}^3$. If, say, $a = 0$, then we drop the expression $(x - x_0)/a$ from the above pair of equations and instead just write $x = x_0$ in addition to the equation $\frac{y - y_0}{b} = \frac{z - z_0}{c}$. If, say, both $a, b = 0$, then the line is just described by $x = x_0, y = y_0, z = t$, where t is any real number.

Examples.

- Consider a line which contains both the points $(2, 3, 5)$ and $(-1, 2, 7)$. Find an equation for this line. We already know a point on this line, so let's find a direction vector. These two points give a direction vector $\vec{v} = \langle 3, 1, -2\rangle$, so this line can be parametrically described by $(2 + 3t, 3 + t, 5 - 2t)$. The second point on the line appears when we let $t = -1$.

  We can also describe this line using the alternate description given above by solving for t. We get
$$t = \frac{x-2}{3} = \frac{y-3}{1} = \frac{z-5}{-2}.$$
- Find a parametric description for the line given by equations
$$\frac{x+2}{3} = \frac{y-1}{4} = \frac{3-z}{2}.$$
  We can solve this problem by either memorizing the relationship between the symmetric form for a line and the parametric form, but even if we don't remember what the exact relationship is, we can still solve this problem by finding two points on this line. For instance, let's find (x, y, z) which makes all three of the above values simultaneously 0. Clearly this is given by $(-2, 1, 3)$. When all three of the above expressions are 1, we get the point $(1, 5, 1)$. A direction vector is then given by $\vec{v} = \langle -3, -4, 2\rangle$, and a parametric equation for the line given by $(-2, 1, 3) + t(-3, -4, 2) = (-2 - 3t, 1 - 4t, 3 + 2t)$.

3. Intersections of lines in $\mathbb{R}^3$

In $\mathbb{R}^2$, two lines are either identical, intersect exactly once, or are parallel to each other and do not intersect at all. What happens in $\mathbb{R}^3$?

It's still the case that if two lines intersect in two different points, then those two lines are identical, since perhaps the defining property of a line is that it is completely determined by any two points on it. So in $\mathbb{R}^3$ we should still expect two lines to either be identical (and hence intersect infinitely often), intersect exactly once, or not intersect.

However, consider the lines $\ell_1 : (t, 0, 0)$ and $\ell_2 : (0, t, 1)$. Clearly these two lines never intersect because their z-coordinates are always distinct. However, they certainly do not describe what we would consider to be parallel lines, because they evidently go in different directions. So perhaps it is possible for two lines to never intersect in $\mathbb{R}^3$ but not be parallel, which is different from what happens in $\mathbb{R}^2$.

In $\mathbb{R}^3$, we say that two lines are *parallel* if they are different lines, and have direction vectors which are scalar multiples of each other. (Notice that in this definition, it does not matter which direction vectors we choose for each of our two lines.) Two lines which never intersect but are not parallel are called *skew*.

Examples.

- Determine whether the two lines $\ell_1 : (3+2t, 3-t, -2+t), \ell_2 : (-1-t, 10+3t, -1+t)$ intersect or not, and if they do not, whether they are parallel or skew.

We need to determine whether there are two values t, s such that $\ell_1(t)$ and $\ell_2(s)$ are equal. In other words, we want to know if there are values of t, s which simultaneously satisfy the equations

$$3 + 2t = -1 - s, 3 - t = 10 + 3s, -2 + t = -1 + s.$$

Notice that it is incorrect to ask whether

$$3 + 2t = -1 - s, 3 - t = 10 + 3t, -2 + t = -1 + t$$

has simultaneous solutions, since perhaps ℓ_1, ℓ_2 intersect at a point where the value of t in ℓ_1 is different than the value of t in ℓ_2.

In any case, back to the original set of equations, we pick any of the three, and solve for s in terms of t (or vice versa, it does not matter). For example, from the first equation we get $s = -4 - 2t$. We then plug this into the other two equations, which eliminates all appearances of s, and then see if any values of t satisfy the remaining two equations simultaneously. In our example, we get

$$3 - t = 10 + 3(-4 - 2t), -2 + t = -1 + (-4 - 2t) \Leftrightarrow 5t = -5, 3t = -3.$$

The latter two equations obviously have simultaneous solution $t = -1$. Solving for s using $s = -4 - 2t$, we find $s = -2$. Plugging either $t = -1$ into the equation for ℓ_1 or $s = -2$ into the equation for ℓ_2, we get a common point $(1, 4, -3)$.

- Determine whether the two lines $\ell_1 : (1 + t, -1 + 2t, 1 + t), \ell_2 : (2 - 2t, 1 + 3t, 4 + t)$ intersect or not, and if they do not, whether they are parallel or skew.

  We try to simultaneously solve

$$1 + t = 2 - 2s, -1 + 2t = 1 + 3s, 1 + t = 4 + s.$$

  The third equation gives $s = t - 3$. Plugging this into the first two, we get

$$1 + t = 2 - 2(t - 3), -1 + 2t = 1 + 3(t - 3) \Leftrightarrow 3t = 7, -t = -7.$$

  Clearly there is no simultaneous solution to this pair of equations, so ℓ_1, ℓ_2 do not intersect. These are skew, and not parallel, because ℓ_1 has $\langle 1, 2, 1\rangle$ for a direction vector, while ℓ_2 has $\langle -2, 3, 1\rangle$ for a direction vector, and these are not scalar multiples of each other.
- Determine whether the two lines $\ell_1 : (t, 1 + t, 3 - 2t), \ell_2 : (1 - 2t, -1 - 2t, 4 + 4t)$ intersect or not, and if they do not, whether they are parallel or skew.

  We try to simultaneously solve

$$t = 1 - 2s, 1 + t = -1 - 2s, 3 - 2t = 4 + 4s.$$

  This time, we will use the first equation which has t in terms of s, and then plug this into the last two equations, replacing all ts with s:

$$1 + (1 - 2s) = -1 - 2s, 3 - 2(1 - 2s) = 4 + 4s \Leftrightarrow 2 = -1, -3 = 0.$$

  These two equations obviously have no simultaneous solutions since $2 = -1, -3 = 0$ aren ever true. So these lines do not intersect. A direction vector for ℓ_1 is $\vec{v}_1 = \langle 1, 1, -2\rangle$, while a direction vector for ℓ_2 is $\vec{v}_2 = \langle -2, -2, 4\rangle$. These are scalar multiples of each other, so these lines are parallel.