MATH 8 CLASS 17 NOTES, 10/29/2010

We saw how the dot product, which takes two vectors and returns a scalar, is a very handy tool for determining the angle between two vectors. The dot product has a natural interpretation in physics, as work, and also allows us to calculate projections of one vector onto another.

We now study another type of vector product, called the cross product, which takes as input two vectors in $\mathbb{R}^3$ and returns another vector in $\mathbb{R}^3$ as output. Unlike the dot product, which is relatively easy to calculate and can be defined for vectors in any $\mathbb{R}^n$, the cross product is unique to $\mathbb{R}^3$ and has an interesting definition.

1. Cross product: Definition and basic properties

Let $\vec{v} = \langle v_1, v_2, v_3\rangle, \vec{w} = \langle w_1, w_2, w_3\rangle$ be two vectors in $\mathbb{R}^3$. Their cross product, $\vec{v} \times \vec{w}$, is defined to be the vector

$$\langle v_2w_3 - v_3w_2, -(v_1w_3 - v_3w_1), v_1w_2 - w_1v_2\rangle$$

What's going on with this definition? Before exploring what geometric properties a cross product has, we will first discuss an interesting way to remember the definition of a cross product using determinants. While this method may seem complicated and artificial, determinants will appear in future linear algebra classes, so exposure to determinants now will acquaint you with them before you learn about them in full generality in linear algebra. As a matter of fact, the fact that you can use determinants to calculate cross products is not coincidental, and can be explained using standard properties of determinants and the geometric properties of cross products. However, we will not say more about this since some knowledge of linear algebra is required.

A matrix is a rectangular array of numbers, with n rows and m columns. We will only be interested in the case of 2×2 and 3×3 matrices, both of which we call square matrices. Given a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we define its determinant to be the number $ad - bc$. We often write $\det A$ for the matrix of A, and sometimes use vertical lines instead of braces to denote the determinant of A. The determinant is a very important number which describes a variety of algebraic and geometric information about its matrix, but we do not have time to discuss those properties in this class.

We now define the determinant of a 3×3 matrix. Given a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

we first define the 2×2 matrix A_{ij} to be the matrix obtained from A by crossing out the ith row and jth column: that is, we cross out the row and column which the entry a_{ij} belongs to. This is called a 2×2 *minor* of A. Then we define the determinant of A to be

$$a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}.$$

A way to remember this is that we need to go along every entry in the top row, multiply that entry by the determinant of its corresponding minor, and then switch signs every time we move to the next entry.

What does this have to do with a cross product? One can check that we can interpret the cross product $\vec{v} \times \vec{w}$ as the determinant of

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

Even though the coordinate vectors $\vec{i}, \vec{j}, \vec{k}$ are not numbers, we can still plug them into the formula for the determinant of a matrix, and obtain a vector. Pay close attention to the order of $\vec{v}, \vec{w}$: the vector which appears first in the cross product MUST be in the second row, not the third row.

Examples.

- Calculate the cross product $\vec{v} \times \vec{w}$ of $\vec{v} = \langle 1, 2, -1 \rangle$ and $\vec{w} = \langle 3, -2, 1 \rangle$. We calculate the determinant of the matrix

$$\det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 3 & -2 & 1 \end{bmatrix} = (2\cdot 1 - (-2)\cdot(-1))\vec{i} - (1\cdot 1 - (-1)\cdot 3)\vec{j} + (1\cdot(-2) - 2\cdot 3)\vec{k} = -4\vec{j} - 8\vec{k}.$$

- Calculate the cross product $\vec{v} \times \vec{w}$ of $\vec{v} = \langle 2, 0, -1 \rangle$ and $\vec{w} = \langle -1, 3, 2 \rangle$. We calculate the determinant of the matrix

$$\det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ -1 & 3 & 2 \end{bmatrix} = (0\cdot 2 - (-1)\cdot(3))\vec{i} - (2\cdot 2 - (-1)\cdot -1)\vec{j} + (2\cdot 3 - 0\cdot -1)\vec{k} = 3\vec{i} - 5\vec{j} + 6\vec{k}.$$

- Calculate the cross products $\vec{i} \times \vec{j}$ and $\vec{j} \times \vec{i}$. Again, we find

$$\det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \vec{k}.$$

  Similarly, we find that $\vec{j} \times \vec{i} = -\vec{k}$. Notice that the cross product is not commutative, and that the order in which we place the terms of the vectors is very important!

This last example illustrates the general fact that $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$. One can check this directly from the definition of the cross product. We also notice that in the last example, $\vec{i} \times \vec{j} = \vec{k}$. Notice that $\vec{k}$ is orthogonal to both $\vec{i}$ and $\vec{j}$. As a matter of fact, this is true in general: the cross product $\vec{v} \times \vec{w}$ is always orthogonal to $\vec{v}$ and $\vec{w}$.

Therefore, you should think of the cross product as a technique which allows you to calculate a vector orthogonal to a given pair of vectors in $\mathbb{R}^3$. There is perhaps one place where you should be careful. If $\vec{v}, \vec{w}$ are scalar multiples of each other (i.e., parallel), then one can check that $\vec{v} \times \vec{w} = \vec{0}$. Of course, $\vec{0}$ is orthogonal to every vector, but is not particularly enlightening if we want to find a nonzero vector orthogonal to a vector in question.

We conclude this section by listing some properties (or non-properties) of the cross product; some are old, and some are new.

- The cross product is *anti-commutative*: $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

- The cross product is not associative: that is, $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$, in general. This means you need to pay close attention to the order in which cross products are taken.
- The cross product is distributive: $\vec{v} \times (\vec{w_1} + \vec{w_2}) = \vec{v} \times \vec{w_1} + \vec{v} \times \vec{w_2}$.
- The cross product commutes with scalar multiplication: $(c\vec{v}) \times \vec{w} = \vec{v} \times (c\vec{w}) = c(\vec{v} \times \vec{w})$.
- The cross product is orthogonal to the two vectors which determine it: $(\vec{v} \times \vec{w}) \cdot \vec{v} = 0, (\vec{v} \times \vec{w}) \cdot \vec{w} = 0$.
- The cross product of two parallel vectors is $\vec{0}$.

2. Geometric interpretations of cross product

As we have seen, the cross product gives a way of producing a vector which is orthogonal to a given pair of vectors. We now want to tackle the question of how to geometrically interpret the direction and magnitude of hte vector $\vec{v} \times \vec{w}$ (we assume that $\vec{v}$ and $\vec{w}$ are not parallel). Indeed, every scalar multiple of this vector is also orthogonal to $\vec{v}$ and $\vec{w}$, so what distinguishes $\vec{v} \times \vec{w}$?

First, we tackle the question of the direction. $\vec{v} \times \vec{w}$ always points in the direction which gives the list of vectors $\vec{v}, \vec{w}, \vec{v} \times \vec{w}$ a *right-handed orientation*. This means that if we take our right hand, curl all the fingers except the thumb in the direction from $\vec{v}$ to $\vec{w}$ (as opposed to the opposite direction), then $\vec{v} \times \vec{w}$ will point in the direction our thumb is pointing in. For example, $\vec{i}, \vec{j}, \vec{k}$ form a right handed orientation. On the other hand (good pun?), $\vec{j}, \vec{i}, \vec{k}$ form a left handed orientation, because if we curl our fingers using our right hand in the direction from $\vec{j}$ to $\vec{i}$, the thumb points not in the direction of $\vec{k}$, but in the direction of $-\vec{k}$. Instead we need to use our left hand to get our thumb in the direction of $\vec{k}$.

What about the magnitude of the cross product? It turns out (we omit the proof here) that the magnitude of $\vec{v} \times \vec{w}$ is equal to the area of the parallelogram determined by $\vec{v}$ and $\vec{w}$! This, in turn, is equal to $||\vec{v}||||\vec{w}|| \sin\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$. In particular, if $\vec{v}, \vec{w}$ are orthogonal to begin with, then $\vec{v} \times \vec{w}$ has norm $||\vec{v}||||\vec{w}||$.

Examples.

- Consider the parallelogram spanned by $\vec{v} = \langle 2, 3, 0 \rangle$ and $\vec{w} = \langle 4, 2, 0 \rangle$. If we think of these two vectors as lying on the chalkboard, in what direction does their cross product point? What is the area of this parallelogram? First, to determine the direction, we use a right hand rule, which tells us that the cross product should point into the board. Next, we calculate the cross product. Since we have lots of 0s in our coordinates this should be relatively easy:

$$\det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ 4 & 2 & 0 \end{bmatrix} = -8\vec{k}.$$

  Notice that this accords with the calculation we made for the direction of $\vec{v} \times \vec{w}$. In any case, this tells us that the area of the parallelogram is 8. We could also have calculated this area by determining $\cos\theta$ using the dot product, and then calculating $||v||||w|| \sin\theta$; however, this is probably more computationally intensive than computing the cross product.
- The two vectors $\vec{v} = \langle 1, 3, -2 \rangle$ and $\vec{w} = \langle 0, -2, 1 \rangle$ determine a plane. (This is the plane passing through the origin and the ends of the two vectors – what conditions must $\vec{v}, \vec{w}$ satisfy for this plane to be unique?) Find an equation that defines this plane.

Recall that we could calculate the equation for such a plane (a plane passing through the origin) if we knew how to find a vector orthogonal to every vector on the plane. However, such a vector would certainly be orthogonal to any two non-parallel vectors on the plane. Therefore, to find such a vector, we should calculate $\vec{v} \times \vec{w}$. We obtain

$$\det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \langle -1, -1, -2 \rangle.$$

Therefore, $\langle -1, -1, -2 \rangle$ is orthogonal to $\vec{v}$ and $\vec{w}$, and one can check that this will actually be orthogonal to every other point in the plane. Therefore, an equation like $x + y + 2z$ defines the plane in question.

3. The scalar triple product

We conclude with a brief discussion about the so-called <u>scalar triple product</u>. Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors in $\mathbb{R}^3$. If they do not all lie on the same plane, then they determine a <u>parallelpiped</u>. Then it is possible to show that the scalar triple product

$$\vec{a} \cdot (\vec{b} \times \vec{c})$$

actually has absolute value equal to the volume of this parallelpiped.