MATH 8 CLASS 16 NOTES, 10/27/2010

We have defined vectors, defined some basic operations on vectors (vector addition and scalar multiplication), have geometric interpretations of these operations, and also talked a bit about the length of a vector. We want to continue our study of the geometry of vectors and three-dimensional space. A critical tool in the study of vectors is an operation known as the *dot product*. Its definition is straightforward, but its applications are surprisingly versatile.

1. Dot product: definition, basic properties

Let $\vec{v}, \vec{w}$ be two vectors in $\mathbb{R}^n$, with coordinates $\langle v_1, \ldots, v_n \rangle, \langle w_1, \ldots, w_n \rangle$. The *dot product* of $\vec{v}$ and $\vec{w}$ is defined to be the real number

$$v_1 w_1 + \ldots + v_n w_n = \vec{v} \cdot \vec{w}.$$

In other words, we just multiply coordinates componentwise and then add them up. The following are some basic properties of dot products, which you can check for yourself using the definition. They basically suggest that the dot product obeys many of the same properties that usual multiplication of numbers obey.

- $(\vec{v}_1 + \vec{v}_2) \cdot \vec{w} = \vec{v}_1 \cdot \vec{w} + \vec{v}_2 \cdot \vec{w}$.
- If c is a scalar, then $(c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w})$.
- $\vec{v} \cdot \vec{v} = |\vec{v}|^2$.

Examples.

- Let $\vec{v} = \langle 2, 3, 5 \rangle$. Then $\vec{v} \cdot \vec{v} = 2^2 + 3^2 + 5^2 = 38$.
- Let $\vec{v} = \langle -2, 5 \rangle, \vec{w} = \langle 5, 2 \rangle$. Then $\vec{v} \cdot \vec{w} = -2 \cdot 5 + 5 \cdot 2 = 0$.
- Let $\vec{v} = \langle 1, 3 \rangle, \vec{w} = -21$. Then $\vec{v} \cdot \vec{w} = 1 \cdot -2 + 3 \cdot 1 = 1$.
- The vector $\vec{u}$ is a unit vector if and only if $\vec{u} \cdot \vec{u} = 1$.

The definition of dot product might seem somewhat arbitrary, but it turns out to have a strong geometric significance. Already we see that taking the dot product of a vector with itself yields the square of its length, which is a geometric property of a vector.

2. Dot product: determining the angle between two vectors

One of the key properties of the dot product is that it provides a way to calculate the angle between two vectors. In particular, we will constantly use the fact that the dot product provides a quick way to determine whether two vectors are perpendicular to each other. (We may also sometimes use the term orthogonal, which in this context is synonymous with perpendicular.)

Consider two vectors $\vec{a}, \vec{b}$, and the third vector $\vec{c} = \vec{a} - \vec{b}$. These three vectors form a triangle; let θ be the angle between the vectors $\vec{a}, \vec{b}$. In particular, $0 < \theta < \pi$.

A formula from geometry, called the law of cosines, which is a generalization of the Pythagorean Theorem, states that

$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos \theta.$$

If we use the fact that $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$, this formula becomes

$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}-2|\vec{a}||\vec{b}|\cos\theta.$$

If we expand the left hand side, we get

$$\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}-2\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}-2|\vec{a}||\vec{b}|\cos\theta.$$

Solving for $\cos\theta$ gives the formula we are interested in:

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}.$$

One can check that this formula holds true even if $\vec{a},\vec{b}$ are collinear, so that $\theta = 0$ or π. It is clear from this formula that $|\vec{a}\cdot\vec{b}| = |\vec{a}||\vec{b}|$ if and only if $\vec{a},\vec{b}$ are scalar multiples of each other, or are collinear.

So the dot product provides a quick way of determining the cosine of the angle between two vectors. In particular, because $\cos\pi/2 = 0$, $\vec{v}\cdot\vec{w} = 0$ if and only if $\vec{v},\vec{w}$ are perpendicular to each other.

Examples.

- Find the angle between the vectors $\langle 3,5,2\rangle$ and $\langle 4,-2,-1\rangle$. We calculate their dot product to be $3\cdot 4+5\cdot(-2)+2\cdot(-1) = 0$, so these vectors are perpendicular to each other.
- Find the angle between the vectors $\langle\sqrt{3},1\rangle$ and $\langle 0,5\rangle$. We calculate their dot product to be 5, which by itself is not enough to answer this question, so we also calculate the norms of these vectors to be 2 and 5. Therefore $\cos\theta = 5/(2\cdot 5) = 1/2$, so the angle between these two vectors is $\pi/3$, or $60°$.
- Suppose $\vec{v},\vec{w}$ are two vectors with length $2,7$ and form an angle of $120°$ between them. What is $\vec{v}\cdot\vec{w}$? We can use the previous formula, in the form $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta$, to find $\vec{v}\cdot\vec{w} = 2\cdot 7\cdot\cos 120° = -7$.

The orthogonality detecting property of the dot product is surprisingly versatile. For example, suppose we want to calculate the distance of a point P, say in $\mathbb{R}^2$, from a line ℓ. For simplicity, let us suppose that ℓ passes through the origin. This distance is the length that P travels if we perpendicularly project P onto the line ℓ. If we think of P as giving the tip of a vector $\vec{v}$, and we think of the line ℓ as containing the nonzero vector $\vec{w}$ (notice that we have lots of different possible choices for $\vec{w}$), then what we want to do is decompose $\vec{v}$ into two pieces, called $\vec{v}^{||}$ and $\vec{v}^{\perp}$, which are two vectors which sum to $\vec{v}$, such that $\vec{v}^{\perp}\cdot\vec{w} = 0$. Then the distance of P from ℓ will just be the length of $\vec{v}^{\perp}$. We sometimes call $\vec{v}^{||}$ the *(vector) projection* of $\vec{v}$ onto $\vec{w}$. Notice that $\vec{v}^{||}$ is actually parallel to $\vec{w}$.

How do we figure out what $\vec{v}^{\perp}$ is from $\vec{v},\vec{w}$? We know that $\vec{v}^{\perp},\vec{v}^{||}$, and $\vec{v}$ form a right-angled triangle with $\vec{v}$ as hypothenuse. Let θ be the angle between $\vec{v}$ and $\vec{v}^{||}$. Because $\vec{w}$ is in the same direction as $\vec{v}$, the angle between $\vec{v}$ and $\vec{v}^{||}$ is the same as the angle between $\vec{v}$ and $\vec{w}$. Then we know that

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}.$$

On the other hand, the length of $\vec{v}^{||}$ is $|\vec{v}|\cos\theta$, which by the above formula equals

$$\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}.$$

This is sometimes called the *scalar projection* of $\vec{v}$ onto $\vec{w}$, and tells you the length of the part of $\vec{v}$ which points in the same direction as $\vec{w}$. Notice that this number can be negative, if $\vec{v}, \vec{w}$ form an angle greater than $90°$.

To find a formula for $\vec{v}^{||}$, all we need to do is multiply the scalar projection (which is a scalar, not a vector) by a unit vector pointing in the same direction as $\vec{w}$. We know how to calculate this; therefore, we get the formula

$$\vec{v}^{||} = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} \cdot \frac{\vec{w}}{|\vec{w}|}.$$

This also gives a formula for $\vec{v}^{\perp}$, since $\vec{v}^{\perp} + \vec{v}^{||} = \vec{v}$. We sometimes write $v^{||}$ as $\text{proj}_{\vec{w}}\vec{v}$, with the notation indicating that we are projecting the vector $\vec{v}$ onto $\vec{w}$.

Examples.

- Suppose we want to project a vector $\vec{v}$ onto a unit vector $\vec{u}$. Because $|\vec{u}| = 1$, the above formula becomes

$$\text{proj}_{\vec{u}}\vec{v} = (\vec{v} \cdot \vec{u})\vec{u},$$

  which evidently is simpler than the more general formula. However, remember, if you want to use this formula, $\vec{u}$ MUST be a unit vector!
- In $\mathbb{R}^2$, the vectors $\vec{i}, \vec{j}$ are the unit vectors $\langle 1, 0\rangle, \langle 0, 1\rangle$, respectively, and one has $\langle x, y\rangle = x\vec{i} + y\vec{j}$. Let's calculate the projection of $\vec{v} = \langle x, y\rangle$ onto $\vec{i}, \vec{j}$.

  In the former case, using the formula from the previous example, $\text{proj}_{\vec{i}}\vec{v} = (\langle x, y\rangle \cdot \langle 1, 0\rangle)\vec{i} = x\vec{i}$, which is what we expect. One can similarly calculate $\text{proj}_{\vec{j}}\vec{v} = y\vec{j}$.

  In $\mathbb{R}^3$, $\vec{i} = \langle 1, 0, 0\rangle, \vec{j} = \langle 0, 1, 0\rangle, \vec{k} = \langle 0, 0, 1\rangle$. Strictly speaking, these $\vec{i}, \vec{j}$ are different from those in the $\mathbb{R}^2$ case, but the context should make it clear what dimensional space $\vec{i}, \vec{j}$ live in should we choose to use this notation.
- Notice that $\text{proj}_{\vec{w}}\vec{v}$ does not change even if we multiply $\vec{w}$ by a nonzero scalar. Indeed, we see that replacing $\vec{w}$ with $c\vec{w}$, where $c \neq 0$, introduces a factor of c^2 in the numerator and a factor of c^2 in the denominator, which end up canceling out.
- Let's check that $\vec{v}^{\perp}$, calculated using the formula $\vec{v}^{\perp} = \vec{v} - \vec{v}^{||}$, really is orthogonal to $\vec{w}$.

  The main tool for detecting orthogonality is to take a dot product and check whether that product equals 0 or not, so let's calculate $\vec{v}^{\perp} \cdot \vec{w}$:

$$\vec{v}^{\perp} \cdot \vec{w} = (\vec{v} - \vec{v}^{||}) \cdot \vec{w} = \vec{v} \cdot \vec{w} - \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} \cdot \frac{\vec{w} \cdot \vec{w}}{|\vec{w}|} = \vec{v} \cdot \vec{w} - \vec{v} \cdot \vec{w} = 0.$$

  So $\vec{v}^{\perp}$ is orthogonal to $\vec{w}$, as claimed!
- Compute the projection of $\vec{v} = \langle 5, 0\rangle$ onto the vector $\vec{w} = \langle 2, 1\rangle$. Using the previous formula for $\text{proj}_{\vec{w}}\vec{v}$,

$$\text{proj}_{\vec{w}}\vec{v} = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2}\vec{w} = \frac{10}{\sqrt{5}^2}\langle 2, 1\rangle = \langle 4, 2\rangle.$$

  We can use this information to calculate the distance from $(5, 0)$ to the line $y = x/2$. Notice that the vector $\langle 2, 1\rangle$ lies on the line $y = x/2$ (ie, its coordinates satisfy $y = x/2$), so that $\text{proj}_{\vec{w}}\vec{v}$ will lie on this line as well. Then we know that $\vec{v}^{\perp} = \langle 5, 0\rangle - \langle 4, 2\rangle = \langle 1, -2\rangle$ is the component of $\vec{v}$ orthogonal to $\vec{w}$, and that the distance from $\vec{v}$ to $y = x/2$ is given by the length of this vector. Since $|\vec{v}^{\perp}| = \sqrt{5}$, the distance of $(5, 0)$ to the line $y = x/2$ is $\sqrt{5}$.

Alternatively, you could have calculated the distance of $(5,0)$ to the line $y = x/2$ using techniques from single variable calculus, by trying to minimize the distance function of $(5,0)$ to a generic point on $y = x/2$. This is a good exercise to do.

- Let's think about how we might describe a plane in $\mathbb{R}^3$ using algebraic equations. Let V be a plane, which for simplicity, passes through the origin. A bit of geometric visualization should convince you that there is exactly one line passing through the origin which is perpendicular to the plane. For instance, if V is the xy plane, then this line would be the z-axis.

  Select a nonzero vector $\vec{v} = \langle a, b, c\rangle$ which is on this line. The fact that this line is perpendicular to V is equivalent to saying that this line is perpendicular to any vector which lies on V, which is equivalent to saying that $\vec{v}$ is orthogonal to any $\vec{x} = \langle x, y, z\rangle$ on V. In terms of the dot product, this means that

  $$\vec{v} \cdot \vec{x} = 0 \Leftrightarrow ax + by + cz = 0.$$

  So the formula for a plane passing through the origin is defined by a single linear equation in the three variables x, y, z, whose coefficients a, b, c are the components of a nonzero vector $\langle a, b, c\rangle$ orthogonal to the plane. We'll come back to questions about planes in a few days.

- Vectors and the dot product have a natural interpretation in physics. Velocity, displacement, acceleration, and force are all expressed using the idea of vectors, because these not only have a magnitude, but also a direction.

  The dot product is well-suited for calculating work. If you have had physics, you might know that work is defined to be force times distance moved, at least if the force is in the same direction as the direction of motion. In contrast, a force which is orthogonal to a direction of motion of an object does no work on that object. More generally, $W = \vec{F} \cdot \vec{x}$, where $\vec{F}$ is force, and $\vec{x}$ is displacement.

  For example, suppose a force (such as a rope someone is pulling) is exerted at a $60°$ angle above the horizontal on an object which is moving in a horizontal direction. If this force has a magnitude of 100 N and the object travels 10 m, then the total amount of work is $\vec{F} \cdot \vec{x}$, which we can calculate using the formula $\vec{F} \cdot \vec{x} = |\vec{F}||\vec{x}|\cos\theta$. In this case, $W = 100 \cdot 10 \cdot 1/2 = 500$ J.