MATH 8 CLASS 15 NOTES, 10/25/2010

For the second half of this class, we begin the study of multivariable calculus. So far, all the ideas in calculus have been applied to functions $f(x)$, which only accept one real variable. We have developed the basic notions of limit, derivative, and integral in this setting, and have seen how these ideas can be applied to a wide variety of problems arising from real life.

Multivariable calculus is the generalization of these ideas to functions of several variables, or perhaps of functions of a single variable which take values in a multi-dimensional space. Of particular interest will be the case of three dimensions, because real life takes place in three spatial dimensions. Much of the language of physics and engineering, for example, relies on concepts from multivariable calculus. The generalization from functions of a single variable to functions of several variables is not entirely trivial. While some of the ideas we encounter will be familiar, there will be some substantially new ideas as well.

Before beginning with multivariable calculus proper, we start by reviewing and developing ideas which allow us to describe multi-dimensional space. Again, of particular interest will be the case of three dimensions.

1. Vectors: Definition and Basic Operations

A fundamental mathematical object in physics and engineering is a vector. A vector can be defined in several ways, all equivalent. In our case, we will define a vector to be an ordered n-tuple of numbers: that is, an element of $\mathbb{R}^n$, which is the set of all n-tuples of real numbers. An alternative definition of a vector is an arrow, of finite length, in a space of dimension n; this is equivalent to our definition if we force all vectors to start at the origin. In this directed arrow representation, the starting point of the origin is sometimes called the initial point, while the endpoint (where the arrow is located) is called the endpoint or terminal point.

There are several different ways to write a vector, all of which are used frequently and often interchangeably. Since a vector is an ordered n-tuple of real numbers, we can just write a vector $\vec{v}$ as $(v_1, v_2, \ldots, v_n)$. The arrow on top of the letter v signifies that we have a vector, although there are times when the arrow is left off. Sometimes, the parentheses are replaced with brackets, like $\langle v_1, \ldots, v_n \rangle$. Finally, in the case of two or three dimensions, a vector (v_1, v_2) or (v_1, v_2, v_3) is sometimes written as $v_1\vec{i} + v_2\vec{j}$, or $v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, respectively. This notation anticipates the fact that we can add vectors and multiply vectors by real numbers, since $\vec{i}, \vec{j}, \vec{k}$ represent $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$ respectively, at least when thought of as in three-dimensional space. This last notation is only used in two and three dimensional space.

When we have various mathematical objects, one question we immediately want to answer is what operations we use on them. In the case of vectors, we have an easy and natural way of defining vector addition and multiplication by a scalar. Suppose $\vec{v} = \langle v_1, \ldots, v_n \rangle$ and $\vec{w} = \langle w_1, \ldots, w_n \rangle$ are two vectors in $\mathbb{R}^n$. Then we can form their vector sum by adding their corresponding components together:

$$\vec{v} + \vec{w} = \langle v_1 + w_1, \ldots, v_n + w_n \rangle.$$

In a similar way, we can define subtraction of two vectors. Vector addition is sometimes graphically depicted by means of the so-called *parallelogram rule*. If we think of $\vec{v}, \vec{w}$ as two directed arrows in $\mathbb{R}^n$, the vector $\vec{v}+\vec{w}$ can be formed by taking the vector $\vec{w}$ and placing its initial point at the endpoint of $\vec{v}$. Then $\vec{v}+\vec{w}$ is the vector which starts at the origin and ends at the endpoint of $\vec{w}$, when translated in this fashion. This is called the parallelogram rule because $\vec{v}$ and $\vec{w}$ form the edges of a parallelogram in this picture, and $\vec{v}+\vec{w}$ is one of the diagonals of this parallelogram.

We can also multiply a vector by a real number. In this context, a real number is often called a scalar, and this operation is called scalar multiplication. Given a real number c and a vector $\vec{v} = \langle v_1, \ldots, v_n \rangle$, we form their product by multiplying each component by c:

$$c\vec{v} = \langle cv_1, \ldots, cv_n \rangle.$$

We skip examples in this section since these operations are not particularly difficult to understand. In physics and engineering, vectors are used to describe many different sorts of concepts. Force, position, velocity, and acceleration are all vectors, for example.

2. Vectors: Norm and Direction

You can think of vector as a mathematical object which contains information about both magnitude (size) and direction. For example, in the arrow picture of a vector, the direction of a vector is the direction which the angle points in, while the magnitude of the vector is its length. We want an algebraic way of describing this information.

Given a vector $\langle v_1, \ldots, v_n$, we define its length, or norm, or magnitude (all of these terms are frequently used), often written $||\vec{v}||$ or $|\vec{v}|$, to be the number

$$\sqrt{v_1^2 + \ldots + v_n^2}$$

This is just the distance of the point $(v_1, \ldots, v_n)$ from the origin, as several applications of the Pythagorean Theorem will tell you. The length is always a nonnegative number, so it encodes no information about the direction a vector is pointing in. For example, if a vector describes the velocity of an object, its length is the speed of that object – hence the dictum that velocity is the speed with a direction attached to it.

We remark that the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$, which arises from a routine application of the Pythagorean Theorem. The distance between two points in higher dimensional same is given by a similar formula.

Of special interest are vectors of length one. These vectors are called unit vectors, and are useful for a variety of reasons. For example, we will find them very helpful when we want to determine the component of a vector lying in a given direction. A frequent calculation we will often come across asks us to find the unit vector pointing in the same direction as a given vector $\vec{v}$. In general, if we have a vector $\vec{v}$, which has length $||\vec{v}||$, then the vector $c\vec{v}$ has length $|c|||\vec{v}||$ – that is, if we multiply a vector by a scalar, its length is multiplied by the absolute value of that scalar. Therefore, to find a vector which has length one and points in the same direction as $\vec{v}$, we simply divide $\vec{v}$ by its own length, to obtain the vector $\vec{v}/||\vec{v}||$.

Examples.

- Calculate the length of $\vec{v} = \langle 3, 12, -4 \rangle$, and find a unit vector which points in the same direction as $\vec{v}$. The formula for length tells us that $||\vec{v}|| = \sqrt{3^2 + 12^2 + (-4)^2} = \sqrt{169} = 13$. Therefore, $\vec{v}$ has length 13, and the unit vector which points in the same direction as $\vec{v}$ is $\vec{v}|||\vec{v}|| = \langle 3/13, 12/13, -4/13 \rangle$.
- If $\vec{v} = \langle v_1, \ldots, v_n \rangle$, what vector is of the same length as $\vec{v}$ and points in the opposite direction of $\vec{v}$? A bit of thought shows that $-\vec{v} = \langle -v_1, \ldots, -v_n \rangle$ is the vector in

question. This is just the vector $\vec{v}$ multiplied by -1; evidently, multiplication by the scalar -1 preserves the length of a vector but reverses its direction.

- Describe the vector which starts at the point $(2,4)$ and ends at the point $(-2,7)$. To calculate this vector, we subtract the initial point from the terminal point, and obtain $\langle -4,3\rangle$. Sometimes it helps to sketch a plot of the points and vector in question.
- Find the two vectors in $\mathbb{R}^2$ which are of unit length and are perpendicular to the vector $\langle 2,-1\rangle$. Even though we haven't yet formally defined the angle between two vectors, in the case of vectors lying in the plane the definition of angle is obvious. If we think of $\langle 2,-1\rangle$ as describing a line passing through the origin and $(2,-1)$, this line has slope $-1/2$. Therefore, the line perpendicular to this line has slope 2, and the vectors we are looking for will lie on this line. We want to find unit vectors which lie on this line; an easy way to do this is to write any vector which lies on the line, such as $\langle 1,2\rangle$, and then divide it by its norm. In this case, we find that $\langle 1/\sqrt{5}, 2/\sqrt{5}$, and its negative, $\langle -1/\sqrt{5}, -2/\sqrt{5}\rangle$, are the two vectors in question.
- Here is a nice trick which sometimes works for calculating the unit vector of vectors which may be somewhat complicated. Consider the vector $\vec{v} = \langle e^{1/3}, 2e^{1/3}, -2e^{1/3}\rangle$. We could use the straightforward way of calculating the unit vector of $\vec{v}$, but the algebra would be somewhat messy because of these $e^{1/3}$ terms. The clever observation to make is that $\vec{v}$ has the same unit vector as the vector $\vec{v}/e^{1/3}$, because $e^{1/3} > 0$ is a positive scalar, and $\vec{v}/e^{1/3} = \langle 1,2,2\rangle$, whose unit vector is easier to calculate. As a matter of fact, in this case, we see that the unit vector of $\vec{v}$ (and $\vec{v}/e^{1/3}$) is $\langle 1/3, 2/3, 2/3\rangle$.

3. Lines and Spheres in three dimensions

We now begin by describing relatively simple geometric objects in three-dimensional space. We start with lines and spheres. In the plane, a line is uniquely determined by specifying any point on the line and its slope. There is no obvious definition of slope in three dimensions, but it is still true that a line is specified by listing a point on the line and a direction the line goes in. (Of course, this is still true in spaces of more than three dimensions).

For example, suppose we were told that $x_0 = (2,1,3)$ was on a line ℓ and that the direction vector $v = \langle 1,-1,2\rangle$ was the direction the line pointed in. Then this line is given by the parametric equation $\ell(t) = (2+t, 1-t, 3+2t) = x_0 + tv$. Notice that the magnitude of the vector v does not actually matter when describing a line this way: the resulting set of points in $\mathbb{R}^3$ will be the same regardless of whether we use v or any scalar multiple of v.

Example. Give a parametric equation for the line passing through the points $(-1,3,2)$ and $(1,1,1)$. We need to find a direction vector for this line; to do so we just subtract one point from the other. For example, $(1,1,1)-(-1,3,2) = (2,-2,-1)$, so $(2,-2,-1)$ is a direction vector for this line, and $f(t) = (-1+2t, 3-2t, 2-t)$ is an equation for this line. Of course, had we chosen the point $(1,1,1)$ to describe the line, we would have obtained a different function but the same set of points.

Recall that a sphere is a set of points all equidistant in $\mathbb{R}^3$ from some fixed point. We call that fixed point the center, and the distance that each of the points on the sphere from the center the radius, which is often denoted r.

For example, the unit sphere S^2 is the set of points in $\mathbb{R}^3$ of distance 1 from the origin. What sort of equation defines a sphere? If (a,b,c) is the center of a sphere, and r is the radius of that sphere, then every point (x,y,z) on the sphere is distance r from (a,b,c). But this set of points is described exactly by the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Examples.

- The unit sphere is given by the equation $x^2 + y^2 + z^2 = 1$.
- What is the equation for a sphere which passes through $(2, -1, 4)$ and has center $(3, 0, 1)$? Since we know what the center is, we need only determine the radius. The radius will be the distance from the center to $(2, -1, 4)$, which is $\sqrt{(3-2)^2 + (0-(-1))^2 + (1-4)^2} = \sqrt{1+1+9} = \sqrt{11}$. Hence the equation for this sphere is $(x-3)^2 + y^2 + (z-1)^2 = 11$.