MATH 8 CLASS 13 NOTES, 10/20/2010

In the last class, we saw how to evaluate integrals with various products of trigonometric functions. We will now consider a special class of substitutions which involve trigonometric functions, which will allow us to evaluate other types of integrals.
Consider the integral $\int \sqrt{1-x^2}\,dx$. None of the methods we currently know will allow us to evaluate this integral. However, if we make a substitution $x = \sin\theta$ (this is perhaps more accurately a reverse substitution), we then have $dx = \cos\theta d\theta$. We restrict θ to lie in the interval $[-\pi/2, \pi/2]$, and our integral becomes

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2\theta}\cos\theta\,d\theta = \int |\cos\theta|\cos\theta\,d\theta = \int \cos^2\theta\,d\theta$$

In the last step, we are allowed to remove the absolute value signs, because we restrict θ to lie in the interval $[-\pi/2, \pi/2]$, in which cos is always nonnegative. We now use our expertise in evaluating trigonometric integrals to calculate

$$\int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

At this point, we need to replace all the θs with xs. To do so, we need to be able to evaluate $\sin 2\theta$. We use the identity $\sin 2\theta = 2\cos\theta\sin\theta$, but this tells us that we need to be able to write $\cos\theta$ in terms of x. Since $x = \sin\theta$, we can think of θ as an angle in a right angled triangle, whose opposite side has length x and hypotenuse has length 1. Then the Pythagorean Theorem tells us that the adjacent side has length $\sqrt{1-x^2}$, and therefore $\cos\theta = \sqrt{1-x^2}$. Therefore, we have

$$\frac{\theta}{2} + \frac{\sin 2\theta}{4} + C = \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} + C = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C$$

More generally, we can handle integrals such as $\int \sqrt{a^2-x^2}\,dx$, by using the substitution $x = a\sin\theta$. Then we obtain

$$\int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2-a^2\sin^2\theta}(a\cos\theta)\,d\theta = \int a^2\cos^2\theta\,d\theta$$

which we can evaluate as before:

$$\int a^2\cos^2\theta\,d\theta = a^2\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C = a^2\left(\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}\right) + C$$

Since $\theta = \arcsin x/a$, to evaluate $\cos\theta$, we draw a right angle triangle with hypotenuse a and opposite side x, and the remaining side has length $\sqrt{a^2-x^2}$. Therefore $\cos\theta = \sqrt{(a^2-x^2)}/a$, and replacing all the θs with xs in the above gives

$$a^2\left(\frac{\arcsin(x/a)}{2} + \frac{x\sqrt{a^2-x^2}}{2a^2}\right) + C$$

Example. Use the integral we calculated above to compute the area of the ellipse defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a, b are positive real numbers. Recall that this equation defines an ellipse with axes of length $2a, 2b$. To calculate the area of this ellipse, we focus on the quarter of the ellipse which lies in the first quadrant. We find that the boundary of the ellipse in the first quadrant is described by the function

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \Rightarrow y = b\sqrt{1 - (x^2/a^2)} = \frac{b}{a}\sqrt{a^2 - x^2}$$

To evaluate the area bounded by this segment of the boundary, we calculate the definite integral of this function from 0 to a:

$$\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

If we use the integral we calculated previously, we find this is equal to

$$\frac{b}{a} \cdot a^2 \left(\frac{\arcsin(x/a)}{2} + \frac{x\sqrt{a^2 - x^2}}{2a^2}\right)\Big|_0^a = \frac{b}{2a}\frac{a^2\pi}{2} = \frac{ab\pi}{4}$$

(Notice that a lot of the terms disappear for $x = 0, a$.) Since we calculated the area of a quarter of the ellipse in question, this shows that the area of an ellipse with axes of length $2a, 2b$ is $ab\pi$. In the case where $a = b = r$, where the ellipse is a circle, we recover the formula $A = \pi r^2$ for the area of a circle.

The method of trigonometric substitution can also be useful if you forget formulas for integrals of arcsin, for instance.

Example. Calculate

$$\int \frac{1}{\sqrt{1 - x^2}}\, dx$$

If we see a $\sqrt{1 - x^2}$, or more generally $\sqrt{a^2 - x^2}$, it is a good bet that trigonometric substitution might work. In this case, take $x = \sin\theta, dx = \cos\theta\, d\theta$, so that the integral in question becomes

$$\int \frac{1}{\sqrt{1 - x^2}}\, dx = \int \frac{1}{\cos\theta}\cos\theta\, d\theta = \int d\theta = \theta + C = \arcsin x + C$$

Here are two other integrals which look like $x = \sin\theta$ is a good substitution.

Examples.

- Evaluate

  $$\int \frac{x}{\sqrt{1 - x^2}}\, dx$$

  We could apply the substitution $x = \sin\theta$, but notice that the simpler u-substitution $u = 1 - x^2$ would work as well, because we have a copy of x available. Even though trigonometric substitution would work in this situation (you should work out this integral using this method), a simpler method is also available. Letting $u = 1 - x^2$, we have $du = -2x\, dx$, so

$$\int \frac{x}{\sqrt{1 - x^2}}\, dx = \int \frac{-1}{2} \cdot \frac{1}{\sqrt{u}}\, du = \frac{-1}{2}2\sqrt{u} + C = -\sqrt{u} + C = -\sqrt{1 - x^2} + C$$

This is considerably less involved than using a trigonometric substitution.

- Evaluate
$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$
This integral looks very similar to the previous integral, but a simple u-substitution would fail here. However, we can still use trigonometric substitution. If we let $x = \sin\theta$, then this integral becomes
$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \int \sin^2\theta = \int \frac{1-\cos 2\theta}{2}\,d\theta$$
If we evaluate this integral and keep track of what $\sin\theta, \cos\theta$ are in terms of x, using the same technique of drawing a right angle triangle as earlier, we obtain the expression
$$\int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} - \frac{(\sin\theta)(\cos\theta)}{2} + C = \frac{1}{2}(\arcsin x - x\sqrt{1-x^2}) + C$$

So far, we have a good idea how to handle an integral like $\int \sqrt{a^2 - x^2}\,dx$. How about the related integrals $\int \sqrt{x^2+a^2}\,dx, \int \sqrt{x^2 - a^2}\,dx$? Just like how we applied the substitution $x = a\sin\theta$ in the original integral to eliminate the square root sign, we can use the identity $\tan^2\theta + 1 = \sec^2\theta$ to eliminate the radicals using an appropriate substitution in the latter two integrals:

Examples.

- Calculate $\int \frac{1}{\sqrt{x^2+a^2}}\,dx$. We make the substitution $x = a\tan\theta$, so that $dx = a\sec^2\theta\,d\theta$. Then the integral becomes
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \int \frac{1}{a\sec\theta}a\sec^2\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$
At this point, we need to convert all the expressions in θ to expressions in x, so we again draw a right angle triangle with suitable lengths. This time, $\tan\theta = x/a$, so we draw a right angle triangle with opposite side x and adjacent side a, so that the hypotenuse has length $\sqrt{x^2+a^2}$. Therefore,
$$\sec\theta = \frac{\sqrt{x^2+a^2}}{a}$$
so we can substitute this into the integral we calculated to obtain
$$\ln\left(\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right) + C$$
(Why are we allowed to remove the absolute value sign?)
- Sometimes we might not see an expression like $\sqrt{x^2-a^2}$, but we could have a very closely related expression like $\sqrt{4x^2-1}$. For example, suppose we wanted to calculate the integral
$$\int \frac{\sqrt{4x^2-1}}{x}\,dx$$
This looks like we should make some sort of substitution like $x = \sec\theta$, although the presence of the 4 will mess things up. Nevertheless, if we instead made the

substitution $x = 1/2 \cdot \sec\theta$, we can get rid of the 4. We have $dx = 1/2 \sec\theta \tan\theta \, d\theta$, so the integral becomes

$$\int \frac{\sqrt{\sec^2\theta - 1}}{1/2 \cdot \sec\theta} \cdot 1/2 \sec\theta \tan\theta \, d\theta = \int \tan^2\theta \, d\theta$$

Notice that we write $\sqrt{\sec^2\theta - 1} = \tan\theta$, without absolute value signs! The reason we are allowed to do this is because we will only let θ range through the values $(0, \pi/2)$ and $(\pi, 3\pi/2)$ in the substitution $x = 1/2 \sec\theta$. This range still covers all the numbers $|x| \geq 1$, while keeping $\tan\theta$ always nonnegative.

We know how to integrate $\tan^2\theta$; we rewrite this integral as

$$\int \sec^2\theta - 1 \, d\theta = \tan\theta - \theta + C$$

Again, we have to reconvert all the expressions in theta to expressions in x. We have $2x = \sec\theta$, so we draw a right angle triangle with hypotenuse of length $2x$ and adjacent side of length 1. Then the opposite side has length $\sqrt{4x^2 - 1}$. In particular, $\tan\theta = \sqrt{4x^2 - 1}$, and $\theta = \operatorname{arcsec} 2x = \arccos 1/(2x)$. (Why is this last equality true?)

Putting this back into our answer, we find that the integral in question is equal to

$$\sqrt{4x^2 - 1} - \arccos 1/(2x) + C$$

Finally, we mention that it is also possible to factor 4 out of the expression $\sqrt{4x^2 - 1}$, to obtain the expression $2\sqrt{x^2 - (1/2)^2}$. Of course, one would still make the substitution $x = 1/2 \cdot \sec\theta$.

We conclude with an example which on the surface does not seem like it can be solved using trigonometric substitution, but using the algebraic technique of "completing the square", does allow us to solve it.

Example. Calculate

$$\int \frac{1}{\sqrt{-3 + 4x - x^2}} \, dx$$

We do not have an expression of $\sqrt{a^2 - x^2}$, or the other two related expressions, in this integral. However, if we 'complete the square' in $-3 + 4x - x^2$, we obtain an expression

$$-3 + 4 - 4 + 4x - x^2 = 1 - (x^2 - 4x + 4) = 1 - (x - 2)^2$$

Therefore, our integral is equal to

$$\int \frac{1}{\sqrt{1 - (x-2)^2}} \, dx$$

This looks a lot like $\sqrt{1 - x^2}$; make a u-substitution $u = x - 2$. We then obtain the integral

$$\int \frac{1}{\sqrt{1 - u^2}} \, du$$

This can be solved using the standard trigonometric substitution techniques, or if you remember, this is also the derivative of $\arcsin u$. Therefore, the integral in question is equal to

$$\arcsin(x - 2) + C$$