MATH 8 CLASS 11 NOTES, 10/15/2010

We will now completely change topics and spend a few classes studying additional integration techniques. Recall that integration is the art of finding an antiderivative of a function. Unlike differentiation, which is relatively 'easy', integration is very hard. For example, it is easy to differentiate a function like $e^x \cos x$, but how do you integrate this function?
We begin with a quick review of u-substitution, which is the integration analogue of the chain rule for differentiation.

1. A review: u-substitution

Recall that the derivative of a composition of functions $f(g(x))$ is $f'(g(x))g'(x)$; this is commonly called the chain rule. Therefore, the integral of $f'(g(x))g'(x)$ will be $f(g(x))+C$. If we are given an integral whose integrand is in the form $f'(g(x))g'(x)$, then we can reduce the integral to (hopefully) a simpler integral, by means of a substitution $u = g(x)$:

$$\int f'(g(x))g'(x)\,dx = \int f'(u)du = f(u) + C = f(g(x)) + C$$

Pay close attention to the change from dx to du, which is obtained by replacing $g'(x)\,dx$ with du. That you should make this replacement is suggested by the Leibniz notation, where

$$\frac{du}{dx} = g'(x)$$

suggests that $'du = g'(x)\,dx$.
To use this rule effectively requires that you be able to recognize derivatives of functions quickly. Unlike differentiation, there is no easy mechanical procedure you can apply to evaluate integrals, and you will have to develop experience by solving many different problems. Let us look at some relatively simple applications of u-substitution.
Examples.

- Evaluate $\int (\cos(\sin x))\cos x\,dx$. In this problem, we see that there is a composite function $\cos(\sin x)$, which strongly suggests that we should make a substitution $u = \sin x$. Furthermore, the fact that there is a $\cos x$ outside the composite function, which is the derivative of $u = \sin x$, confirms the intuition that we should use u-substitution. Since $u = \sin x, du = \cos x\,dx$. Using u-substitution yields

$$\int (\cos(\sin x))\cos x\,dx = \int \cos u\,du = \sin u + C = \sin(\sin x) + C$$

  At the end of the problem, make sure to replace all the us with $g(x)$, since the initial integral is phrased in terms of the variable x.
- Evaluate $\int xe^{-x^2}\,dx$. Again, we see an $-x^2$ in the exponent of an exponential, which strongly suggests that we should try $u = -x^2$. (One could also use $u = x^2$.) Then $du = -2x\,dx$, so using the u-substitution yields

$$\int xe^{-x^2}\,dx = \int -\frac{e^u}{2}\,du = -\frac{e^u}{2} + C = -\frac{e^{-x^2}}{2} + C$$

Notice that if the integrand were instead e^{-x^2} instead of xe^{-x^2}, then u-substitution would fail, and as a matter of fact, it is not possible to integrate e^{-x^2} in terms of any functions we are familiar with.

- Sometimes finding the correct u-substitution can be non-trivial. Evaluate the integral

$$\int \frac{1}{t \log t}\, dt$$

For this problem, there does not seem to be an obvious u to choose. However, we probably should choose either $u = 1/t$ or $u = \log t$, since these are the only components of the integral that we can really see. Let's choose $u = \log t$. Then $du = dt/t$, so the integral becomes

$$\int \frac{1}{t \log t}\, dt = \int \frac{du}{u} = \log u + C = \log(\log t) + C$$

2. Integration by Parts

One of the most versatile integration techniques is known as integration by parts. Just like how u-substitution arises from the chain rule for derivatives, integration by parts arises from the product rule for derivatives. Recall that the product rule says that $(fg)' = f'g + fg'$. If we integrate both sides, we have an equation

$$\int (f(x)g(x))'\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx$$

We can evaluate the integral on the left, and if we isolate the second term on the right, we have the equation

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$$

This might not seem like we have done anything new, but in actuality this formula, which is the <u>integration by parts</u> formula, is a powerful new integration technique that will allow us to compute integrals that previously we were unable to solve.
A common practice is to write $u = f(x), dv = g'(x)\, dx$, and so write $du = f'(x), v = g(x)$. Then the integration by parts formula looks like

$$\int u\, dv = uv - \int v\, du$$

The key feature of the integration by parts formula is that it allows us to take an integral whose integrand is the product of two functions $f(x), g'(x)$, and replace the problem of solving that integral by solving an integral whose integrand is $f'(x), g(x)$. In other words, we can transform any integrand which is the product of two functions into another where we differentiate one of the terms and integrate the other term. Even though it seems like we will not get anywhere when we do this (after all, a derivative on one part of the integral has to be balanced with an integral on the other), there are many situations where integrating does not increase the 'complexity' of a function.

Example. A standard example for integration by parts is the integral $\int xe^x\, dx$. There is no u-substitution that would let us solve this integral. However, we see that this integral is the product of two functions, x, e^x. If we differentiate x, we end up with the simpler function 1, while integrating e^x still gives us e^x. Therefore, the integration by parts formula, with $u = x, dv = e^x\, dx$, gives $du = dx, v = e^x$, and therefore

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Sometimes a problem requires multiple integration by parts to solve.

Example. Compute the integral $\int x^2e^x\,dx$. We should apply integration by parts to $u = x^2, dv = e^x\,dx$, and therefore $du = 2x\,dx, v = e^x$. Then integration by parts yields

$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx$$

At this point, if we had not done the above example, we would have to apply integration by parts again, to the integral on the right hand side. But since we have already calculated this integral, we can substitute that answer to obtain

$$\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$$

From this example, it is clear that one could use integration by parts n times to integrate $\int x^ne^x\,dx$, with each use of the integration by parts formula reducing the power of x by one.

There will be times when integration by parts seems to not get very far, but actually does end up being useful. The following example illustrates a common trick when integration by parts involves sin or cos.

Example. Integrate $\int e^x\cos x\,dx$. Both e^x and $\cos x$ are easy to integrate and differentiate, so let's choose $u = e^x, dv = \cos x\,dx$. Then $du = e^x\,dx, v = \sin x$. One application of integration by parts yields

$$\int e^x\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx$$

However, $e^x\sin x$ isn't any easier to integrate than $e^x\cos x$. Nevertheless, we can try integration by parts again on $e^x\sin x$. This time, let $u = e^x, dv = \sin x\,dx$. (We make this choice of u, dv because if we chose $u = \sin x, dv = e^x\,dx$, we would end up back at the original problem again.) Then $du = e^x\,dx, v = -\cos x\,dx$. Then we can write the above equation as

$$\int e^x\cos x\,dx = e^x\sin x - (-e^x\cos x + \int e^x\cos x\,dx)$$

Even though it looks like we just got our original integral back, something non-trivial happened. Namely, we can actually add $\int e^x\cos x\,dx$ to both sides, and we obtain

$$2\int e^x\cos x\,dx = e^x\sin x + e^x\cos x + C$$

or

$$\int e^x\cos x\,dx = \frac{1}{2}(e^x\sin x + e^x\cos x) + C$$

One could apply the same method to evaluate $\int e^x\sin x\,dx$.

Sometimes integration by parts is not so obvious to apply. In particular, there may be times where you need to let $dv = dx$.

Examples.

- Integrate
$$\int \ln(x^2+1)\,dx$$
After some brief consideration, it is clear that u-substitution isn't going to work. Instead, let's try integration by parts. We can't integrate $\ln(x^2+1)$, so we want to set that equal to u. Since there isn't anything left in the integrand, we have to let $dv = dx$. Then we have
$$du = \frac{2x\,dx}{x^2+1}, v = x$$
The integration by parts formula yields
$$\int \ln(x^2+1)\,dx = x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\,dx$$
The latter integral is something we can evaluate. If we eliminate the power of x^2 in the numerator, we obtain
$$\int \frac{x^2}{x^2+1}\,dx = \int 1 - \frac{1}{x^2+1}\,dx = x - \arctan x + C$$
Therefore, the original integral we wanted to evaluate is equal to
$$\int \ln(x^2+1)\,dx = x\ln(x^2+1) - 2x + 2\arctan x + C$$
- Integrate
$$\int \arctan x\,dx$$
Again, it's not clear how to do this integral, so we will try integration by parts. Since we don't know how to integrate $\arctan x$ yet, we set $u = \arctan x, dv = dx$. Then
$$du = \frac{1}{x^2+1}, v = x$$
and the integration by parts formula yields
$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{x^2+1}\,dx$$
This latter integral is something we can evaluate, using u-substitution. If we let $u = x^2+1$, some quick calculations shows that we have
$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\log(x^2+1) + C$$

Integration by parts can certainly be a difficult integration technique to use. It is quite computationally intensive (you need to be able to at least differentiate and integrate simpler parts of the integrand), and in some cases you have to apply integration by parts many times, which requires you to keep track of signs and constants accurately. However, integration by parts also greatly increases the number of integrals you will be able to solve. As always, practice is the key to learning and becoming comfortable with this powerful integration method.