Math 8 Preparation

1. Algebra and Trigonometry

In Stewart: §7.6 and Appendices A and D.

1.1. **Absolute Value.** This is the only topic in our brief algebra review. The absolute value of a number is just its magnitude, its distance from 0 if we plot them on a number line. We denote it with vertical lines: $|5| = 5$, $|-2| = 2$. The key idea I would like to refresh you on is that $|x| < c$ (where c is positive) expands to $-c < x < c$, and likewise for $\leq$ (where c could also be 0).

Exercise. Simplify the following expressions so that absolute value is applied to as little as possible, and remove any extraneous negations.

(1) $\left|\dfrac{(-1)^{2n+1}}{8(n-5)}\right|$.

(2) $\left|\dfrac{-x^2}{4(n+10)^2}\right|$.

(3) $\left|\dfrac{-3(n+2)}{n^3}\right|$.

Can you simplify further if we specify $n > 0$? If $x > 0$?

Exercise. Solve the following equations and inequalities.

(1) $\dfrac{|x+3|}{2} < 1$.

(2) $\left|\dfrac{2x-1}{x+1}\right| = 3$.

(3) $\dfrac{-1}{|2x-1|} > -2$.

1.2. **Angles and Identities.** Trigonometry shows up heavily in the techniques of integration we'll cover. It will be good for you to know your reference angles, even if just as a time saver.

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$
0	0	0	1
30	$\pi/6$	$1/2$	$\sqrt{3}/2$
45	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
60	$\pi/3$	$\sqrt{3}/2$	$1/2$
90	$\pi/2$	1	0

Important identities:

(i) $\sin^2\theta + \cos^2\theta = 1$
(ii) $\tan^2\theta + 1 = \sec^2\theta$
(iii) $\sin(2\theta) = 2\sin\theta\cos\theta$
(iv) $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$
(v) $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$

You will need to have these all by heart during the course. Notice you can get (ii) from (i) by dividing through by $\cos^2\theta$. In fact many of these may be derived from each other and your book's trigonometry review shows you how to do so, but (except for the relationship between (i) and (ii)) it may be easier just to memorize them.

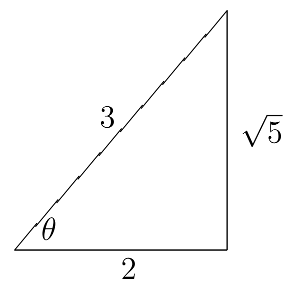
One of the things we will do with these identities is simplify complicated trig expressions into nicer ones, where the "niceness" criterion has to do with the size of the exponents and the opportunity for integration by substitution.

Exercise. Get from point A to point B in each problem below, using the identities above.

	Point A	Point B
(1)	$\sin^5 x\cos^2 x$	$(\cos^2 x - 2\cos^4 x + \cos^6 x)\sin x$
(2)	$\sin^4\theta$	$\frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$
(3)	$\tan^5 x\sec^3 x$	$\tan x(\sec^7 x - 2\sec^5 x + \sec^3 x)$
(4)	$\cos^3\theta\sin(2\theta)$	$2\sin\theta\cos^4\theta$

1.3. **Inverse Trigonometric Functions.** Now recall inverse trig functions, notated with a superscript -1 or the prefix "arc". If $\arcsin x = \theta$, then $\sin\theta = x$. Most inverse trig functions have limited domains, but knowing those won't be vital to our purposes so we'll ignore them.

Suppose you want to simplify $\sin(\arccos(\frac{2}{3}))$. You could use a calculator, I suppose, if you must (though not on an exam), but you can also use a triangle, which will give you a more exact answer. There is some angle θ such that $\arccos(\frac{2}{3}) = \theta$, and we want to find $\sin\theta$. Since cosine is the ratio of lengths of the leg adjacent to θ and the hypotenuse, we know a triangle where those lengths are 2 and 3 respectively will give the correct θ – and since all such triangles will be proportional and all that matters is the *ratio* of the lengths, filling in the third side will suffice to find the sine.



We conclude $\sin\theta = \frac{\sqrt{5}}{2}$.

Exercise. Simplify each of the following expressions, without a calculator.

(1) $\cos(\sin^{-1}(2/3))$
(2) $\sec(\arctan(3))$
(3) $\tan(\sec^{-1}(x/2))$
(4) $\sin(\arccos(x+1))$

2. Limits

In Stewart: §2.3, 4.4, and 7.8.

2.1. **Limit Laws.** You will want to be very comfortable with limits and their manipulation, because they show up in several contexts.

First, recall the limit laws. Suppose c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then:

(1) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$.
(2) $\lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x)$.
(3) $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
(4) If $\lim_{x\to a} g(x) \neq 0$, $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

Using these laws, along with the rules $\lim_{x\to a} x = a$ and $\lim_{x\to a} c = c$, we can draw further conclusions. For example, it is legal to pull powers outside of limits (that is, $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, and similarly for roots, though that requires extra proof).

Note carefully that **you may only take limits apart using the limit laws if all component limits exist.** If you take the limit apart and you get, say, a zero on the bottom of a fraction, or something that limits to infinity, you may not conclude anything about the original limit. The limit laws allow you to put together convergent limits into convergent limits; they do *not* allow you to claim divergence based on divergence of pieces.

In finding the following limits, remember that $\lim_{x \to a} f(x)$ *explicitly ignores* the actual value (or lack thereof) of f at a. Hence any algebraic manipulation (such as cancellation: remember *removable discontinuities*) which gives a function equal to f except possibly at a gives a function with limit at a equal to f's limit at a.

You may also wish to review one-sided limits on your own; the key fact is that even if the one-sided limits at a both exist for a function f, if they are not equal the (two-sided) limit of f at a does not exist.

Exercise. Find each of the following limits.

(1) $\lim_{t \to 4} \dfrac{4-t}{2-\sqrt{t}}$.

(2) $\lim_{h \to 0} \dfrac{(2+h)^2 - 4}{h}$.

(3) $\lim_{x \to 0} \dfrac{\sqrt{3+x} - \sqrt{3}}{x}$.

2.2. **Limits at Infinity.** We will heavily use limits of the form $\lim_{x \to \infty} f(x)$.

There are three ways a continuous function f can behave as we move out along the positive x-axis: it can approach a finite value (have a horizontal asymptote), approach $\pm\infty$ (i.e., have its magnitude grow unboundedly), or oscillate.

If f is a rational function – a quotient of polynomials – then it will never oscillate as x approaches ∞, but its options are otherwise not limited. The behavior of f depends on the ratio of its leading powers. The rule is as follows:

Suppose $f(x) = p(x)/q(x)$ where p is a polynomial of degree n and q a polynomial of degree m. Then if $n > m$, $\lim_{x \to \infty} f(x)$ is infinite (positive if the leading coefficients of p and q have the same sign, and negative otherwise). If $n < m$, $\lim_{x \to \infty} f(x) = 0$, and if $n = m$, $\lim_{x \to \infty} f(x)$ equals the ratio of leading coefficients of p and q.

The proof of this involves l'Hospital's rule for evaluating limits at infinity of indeterminate forms. The rule states that if $\lim_{x \to \infty} f(x)$ and $\lim_{x \to \infty} g(x)$ are both 0 or both $\pm\infty$, then $\lim_{x \to \infty} f(x)/g(x) = \lim_{x \to \infty} f'(x)/g'(x)$.

Example. As an example of l'Hospital's rule and the rational function rule, let's find the limit

$$\lim_{x\to\infty} \frac{x^2+x-2}{2x^2-3x+1}.$$

The rational function rule tells is this limit is $1/2$, because the two polynomials are both degree 2 and the leading coefficients are 1 on top and 2 on the bottom. L'Hospital's rule applies to this because both polynomials approach positive infinity as x does. Hence

$$\lim_{x\to\infty} \frac{x^2+x-2}{2x^2-3x+1} = \lim_{x\to\infty} \frac{2x+1}{4x-3}.$$

We still have the indeterminate form ∞/∞, so apply l'Hospital once more:

$$\lim_{x\to\infty} \frac{2x+1}{4x-3} = \lim_{x\to\infty} \frac{2}{4} = \frac{1}{2}.$$

An intuition about *growth rates*, which comes out of l'Hospital's rule, will be very useful in this class. What happens when you take derivatives of various functions can tell you how they behave in the limit when you make fractions out of them.

Natural log, $\ln x$, has an infinite limit at infinity, but it grows very slowly. You can see, in fact, that when you take its derivative and get $1/x$, growth has been contributed to the opposite side of the fraction:

$$\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x.$$

Polynomials don't work against their own side of the fraction like natural log does, but they are whittled away by a finite number of derivatives, so they are in the middle as far as growth rates go.

Exponentials, e^x and related functions, survive differentiation virtually unchanged, and trump all standard continuous functions. As an exercise, try finding $\lim\limits_{x\to\infty} e^x/x^3$ using l'Hospital's rule (that is, the long way – the short way is to say "e^x grows much faster than x^3 and hence this limit is infinity." There will be times when that is sufficient, times when all you're looking for is guidance on what other method to apply to the problem).

Of course I have skipped many functions, such as square roots, but this provides a starting point. It is clear how one compares square roots to polynomials and exponentials, and less clear how they ought to relate to natural log. We'll explore that a little in the exercises.

Exercise. For each of the following functions, guess the value of the limit as x approaches infinity and then use the limit laws, l'Hospital's rule and the rational function rule to find it explicitly.

(1) $\dfrac{\ln x}{\ln x^2}$

(2) $\dfrac{\sqrt{x}}{\ln x}$

(3) $\dfrac{3x-4}{\sqrt[3]{x^4+x^2-1}}$

3. Derivatives and Integrals

In Stewart: throughout, but particularly §3.9 and 5.5.

3.1. **Linear Approximation and Differentials.** I expect by now differentiation is nearly second nature for you, having done it for a long time. Hence the only review we'll do here is of linear approximation, or in other words using a tangent line as an approximation to a function and all the associated vocabulary and values.

Linear approximation is just a name applied to the tangent line when we wish to think of it as a function in its own right, used as a simplification of the original function, which should be relatively accurate as such near the point of tangency. If we take $f(x)$ and approximate it at a, the point-slope line formula $y - y_0 = m(x - x_0)$ becomes $y - f(a) = f'(a)(x - a)$. When thinking of y as a function of x we might call it $L(x)$, the *linearization* of f at a.

Now we have two functions, $f(x)$ and $L(x)$, which are theoretically giving us similar output values when we input x values near a. Therefore we can compare them, which we do via *differentials*. At a, f and L are equal. As the value of x moves away from a, in most cases they become unequal. By how much do they differ? Let's do this via an example and then the generality.

Example. Let $f(x) = x^3 - x + 1$ and find the linearization of f at 2.

We need several pieces: $f(2) = 2^3 - 2 + 1 = 7$ is our y_0 for the tangent line. The slope of the tangent line is $f'(2)$, where $f'(x) = 3x^2 - 1$, so it is 11. And of course the x_0 is 2 itself.

$$L(x) = 7 + 11(x - 2) = 11x - 15.$$

Now consider $x = 3$. $f(3) = 3^3 - 3 + 1 = 25$ and $L(3) = 33 - 15 = 17$. They differ by 8, but what that difference means for L as an approximation depends on how far away from the point of tangency we had to go to get that difference. In this case, we went one unit right; we notate this by $dx = 1$ and $\Delta x = 1$. The d goes with the approximation and the Δ with the original function, but for input they are always equal so as to have a meaningful comparison. At the point of tangency, $x = 2$, the output value was 7; at $x = 3$ for f it had increased to 25, for a difference in y of 18, which we notate by $\Delta y = 18$. For L the change in y was 10, denoted $dy = 10$.

In general, we call dx and dy *differentials*, distinguishing them from the actual change given by Δx and Δy. Note that because $L(x) = f(a) + f'(a)(x - a)$ and $dy = L(x) - f(a)$, we can compactly give $dy = f'(a)(x - a)$. Furthermore, since $x - a$ is exactly dx, we get $dy = f'(a)dx$. This should make sense, as the approximation is linear.

Exercise. Find $L(x)$, dy and Δy for $f(x) = 3x^2 - 2$, $a = 1/2$, and (first) $dx = \Delta x = 1$ and (second) $dx = \Delta x = -1$.

3.2. **Integration.** Besides being able to integrate polynomials, you should immediately be able to integrate all the following functions:

- e^x, $1/x$
- $\sin x$, $\cos x$, $\sec^2 x$, $\sec x \tan x$
- $1/(1 + x^2)$

The last one in particular will crop up in settings where you expect to see things that require more specialized techniques. The integrals above, of course, will usually appear in forms that require substitution to integrate.

Let's do one example, because manipulating $u = f(x)$ by more than constant multiples may not be something you're accustomed to.

Example. Find the integral

$$\int \frac{x}{\sqrt[3]{2x+1}} dx.$$

Ordinarily when you see some function of x under a radical you can let it be u, you find the rest of it is du up to a constant multiple, and you're home free. Here, however, we have this x on top of the fraction, which is not part of du. What to do? We still let $u = 2x + 1$, noting $du = 2dx$, but for the x on top we solve $u = 2x+1$ for x: $x = \frac{1}{2}(u-1)$. Having faith this will all work out somehow, substitute everything back into the original integral:

$$\int \frac{\frac{1}{2}(u-1)}{\sqrt[3]{u}} \left(\frac{1}{2} du\right).$$

This cleans up nicely to something we know how to integrate directly:

$$\begin{aligned}\frac{1}{4}\int\left(u^{\frac{2}{3}}-u^{-\frac{1}{3}}\right)du&=\frac{1}{4}\left[\frac{3}{5}u^{\frac{5}{3}}-\frac{3}{2}u^{\frac{2}{3}}\right]+C\\&=\frac{1}{4}\left[\frac{3}{5}(2x+1)^{\frac{5}{3}}-\frac{3}{2}(2x+1)^{\frac{2}{3}}\right]+C.\end{aligned}$$

This can be simplified a little with algebra, but not much.

Exercise. Find each of the following integrals.

(1) $\displaystyle\int_{-1}^{0}\frac{1}{(2t+3)^{3.2}}\,dt$

(2) $\displaystyle\int\sec^3\theta\tan\theta\,d\theta$

(3) $\displaystyle\int\frac{dx}{\sqrt{x}(1+x)}$

(4) $\displaystyle\int_0^{\pi/2}\frac{\cos\theta}{1+\sin\theta}\,d\theta$

(5) $\displaystyle\int\frac{dt}{\cos^2 t\sqrt{1+\tan t}}$

(6) $\displaystyle\int x^3\sqrt{x^2+1}\,dx$

(7) $\displaystyle\int_1^{\sqrt{3}}\frac{\sqrt{\tan^{-1}x}}{1+x^2}\,dx$

4. Sequences and Series

In Stewart: Chapter 12.

A *sequence* is an infinite (ordered) list of numbers. A *series* is a sum of infinitely-many numbers. This is often considered the most difficult part of the course, in part because it bears the least resemblance to the topics in a first course in calculus. Unlike the others, this section is a preview rather than a review, to try to build some intuition and fluidity.

We don't spend a lot of time on sequences in the class, but they are a little easier to get a handle on so we'll discuss them a bit now.

There are various notations for sequences, from simply listing the values $a_0, a_1, a_2, \ldots$ to the more compact $\{a_n\}_{n=0}^{\infty}$ to the yet more compact $\{a_n\}$. The subscript n is called the *index* and is simply a counter. Sometimes we think of sequences as being discrete values of a function (say f) taking the index as its input, and in that case we might notate as $\{f(n)\}$, with or without the $n = 0$ and ∞ (I should note the book begins sequences and series with $n = 1$ in the early part of the chapter, and midway through switches to $n = 0$; as long as f matches up to give the correct values the indexing does not matter).

Exercises.

(1) $0, 2, 4, 6, \ldots$ could be written $\{2n\}_{n=0}^{\infty}$. How might you write it to start with index $n = 1$ but be the same sequence?
(2) Write functions to give the sequence $9, 16, 25, 36, \ldots$ with the index starting at $n = 1$ and at a second value of your choosing.
(3) $f(n) = (n + 1)^2$ and $g(n) = n^2 + 8n + 16$ give the same sequence – provided the indices start at the correct places. If $\{f(n)\}_{n=k}^{\infty}$ and $\{g(n)\}_{n=\ell}^{\infty}$ are giving the same sequence, what is the relationship between k and ℓ?
(4) Suppose we modify the sequence in (1) so its terms are alternating in sign. How could you modify $2n$ to give the new sequence $0, -2, 4, -6, 8, \ldots$?

The main, or possibly only, distinction we make in sequences is whether or not they *converge*. Convergence is having a limit: it means there is a finite value L such that for any $\varepsilon > 0$, there is some N such that every term of the sequence from a_N on is within ε of L. None of the sequences given as examples above converge – the first three head off to positive infinity and the fourth bounces around wildly towards both positive and negative infinity. We say they *diverge*.

One thing that takes getting used to when working with sequences and series is that we have a lot of one-way implications. That is, there are a lot of statements of the form "If condition A holds of the sequence, then it converges." It is important to remember if condition A *fails*, this statement says *nothing* about the convergence or divergence of the sequence – we say "the test fails." As an analogy, think about the statement "If the integer n is divisible by 6, then it is even." If I give you an integer to test for evenness and tell you it is divisible by 6, you are done. If I tell you it is not divisible by 6, though, you can draw no conclusion: odd and even numbers may both be nondivisible by 6, such as 7 and 14.

Let me introduce some vocabulary. A sequence is *increasing* if every term is greater than the previous: for all n, $a_{n+1} > a_n$. It is *decreasing* if the opposite inequality holds, and *monotone* if it is either increasing or decreasing. The first three sequences above are increasing and the fourth is not monotone.

Think for a little while about whether monotone sequences can converge or not. Draw pictures.

A sequence is *bounded above* if there is some m such that for all n $a_n \leq m$. It is *bounded below* if the reverse inequality holds for some m, and *bounded* if it is bounded *both* above and below. What kinds of sequences are bounded? What is the relationship between boundedness and monotonicity (if any)?

As analogies, consider the limit at infinity of the following functions:

- $f(x) = x$
- $g(x) = 0$
- $h(x) = \sin x$
- $r(x) = (\cos x)/x$
- $s(x) = 1/x$
- $t(x) = (x-1)/(x+1)$

We'll leave off with sequences now and go on to series. Most of the relevant chapter of the book is on series, as they are more complicated to work with than sequences. First the notation: the infinite sum $a_0 + a_1 + a_2 + \ldots + a_n + \ldots$ will be notated

$$\sum_{n=0}^{\infty} a_n.$$

Sometimes the $n = 0$ and ∞ will be left off, but not often.

Here, convergence means the sum is finite. Think of a series as a sort of infinite Riemann sum, representing every term of the series as a rectangle of that area. If the rectangles don't shrink down, our sum will have to be infinite (think of adding infinitely-many 1s). In fact, the rectangles have to shrink very rapidly as we move off to infinity, so the area that remains to be added not only decreases but drops off drastically as we go along.

This gives us the first test we have for series, called the Test for Divergence. If the terms of the series don't converge to 0 (as a sequence), then the series can't converge.

Exercise. What conclusions can you draw about the convergence of the following series?

(1) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-3}{n^2+1}$.

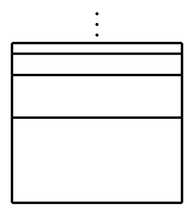
(3) $\sum_{n=1}^{\infty} n^{1/5}$.

So when a series makes it through the Test for Divergence, what then? Let's look at two examples pictorially.

First, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

If we make rectangles of width 1 with area given by the terms of the series, we can stack them up:

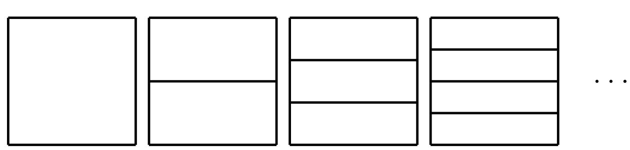


It should be apparent that this series converges, because (if the bottom of the stack is at 0) every additional rectangle covers only half the distance between the top of current stack and 1. Can you prove the series sums to exactly 1? (How would you do that? This is one of the very few series where finding the actual *value* of the sum is feasible.)

Next, look at the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \ldots.$$

The Test for Divergence says nothing about this series, because the terms are in fact converging to 0. But does the series converge? It should be clear that it does not. Stacking rectangles shows us this series is actually $\sum 1$ in disguise:



The cut-off point, of course, is somewhere in-between. You don't have to shrink as fast as $\frac{1}{2^n}$ to converge, and you don't have to shrink as slowly as the second example to diverge. The turnover happens close to $\frac{1}{n}$. We will see that $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n \ln n}$ diverges, but $\sum \frac{1}{n^{1.0000001}}$

converges. This is where intuition about growth rates will be advantageous. We'll leave off here.

5. Hints

§1.1:
You need the rules $|ab| = |a| \cdot |b|$, $|a^n| = |a|^n$, and $a > b \Longleftrightarrow -a < -b$. Remember exponentiation is done before negation, so $-x^2$ is equivalent to $-(x^2)$. For (3) in the second exercise, the opposite expansion to the one mentioned before the first exercise will apply: $|x| > a \Longleftrightarrow x > a$ or $x < -a$.

§1.2:
Try every identity that applies, and remember you can think of, e.g., $\sin^3 x$ as $\sin^2 x \sin x$. Working backwards sometimes helps.

§1.3:
Rewrite 3 as 3/1; treat x the same as a number; remember you cannot cut up the sum inside the arccos.

§2.1:
(1): There are two ways to do this but both involve $2 + \sqrt{t}$ somehow.
(2): explore simple algebraic maneuvers.
(3): can you multiply by 1 in a useful form, to clean up the numerator?

§2.2:
All of these have the form ∞/∞; remember since you're taking the limit as x approaches positive infinity, you may assume x is as large as you like. You don't actually need l'Hospital for (1) – unless you've forgotten some logarithm-manipulation rules. For intuition purposes, in (3) ignore everything but the leading term under the radical. L'Hospital's rule is not useful for (3) because the radical will keep spitting out polynomials when you take its derivative. Try multiplying through top and bottom by $x^{-4/3}$ instead, and apply the limit laws.

§3.1:
There is not much to offer in the way of hints for this one. Just be sure for the second value of Δx you're going from $x = 1/2$ to $x = -1/2$.

§3.2:
(1) is substitution you can probably do in your head.
(2) might have the cluster of secants profitably broken up.
(3) asks you to think of x as a power of something.
(4) should be straightforward.
(5) will be straightforward once you take one of the trig functions out

of its disguise.
(6) needs the approach of the example.
(7) is just a single substitution, with only two reasonable possibilities.

§4:
Mainly, don't overthink these. What is the simplest function that could work out? For (4) in the first set of exercises you may find it helpful to think about the very first exercise of §1.1.

6. Solutions

6.1. **Final Answer Only.** Exercises for which this makes no sense are skipped.

§1.1

(1) $\dfrac{1}{8|n-5|}$, no further simplification possible.

(2) $\dfrac{x^2}{4(n+10)^2}$, no further simplification possible.

(3) $\dfrac{3|n+2|}{|n|^3}$; if $n>0$, simplifies to $\dfrac{3(n+2)}{n^3}$.

(1) $-5<x<-1$.

(2) $x=-4$ or $x=-2/5$.

(3) $x>3/4$ or $x<1/4$.

§1.3

(1) $\sqrt{5}/3$.
(2) $\sqrt{10}$.
(3) $\frac{1}{2}\sqrt{x^2-4}$.
(4) $\sqrt{-(x^2+2x)}$.

§2.1

(1) 4.
(2) 4.
(3) $\frac{1}{2\sqrt{3}}$.

§2.2

(1) 1/2.
(2) ∞.
(3) 0.

§3.1
$L(x) = 3x - \frac{11}{4}$.
For $dx = \Delta x = 1$, $\Delta y = 6$ and $dy = 3$.
For $dx = \Delta x = -1$, $\Delta y = 0$ and $dy = -3$.

§3.2

(1) $\dfrac{1}{4.4}\left(1 - 3^{-2.2}\right)$.

(2) $\frac{1}{3}\sec^3\theta + C$.

(3) $2\arctan\sqrt{x} + C$.

(4) $\ln 2$.

(5) $2\sqrt{1+\tan t} + C$.

(6) $\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C$.

(7) $\dfrac{(8-3\sqrt{3})\pi^{3/2}}{36\sqrt{3}}$.

§4

(1) $\{2n-2\}_{n=1}^{\infty}$.
(2) $\{(n+2)^2\}_{n=1}^{\infty}$, $\{n^2\}_{n=3}^{\infty}$.
(3) $k = \ell + 3$.
(4) $\{(-1)^n(2n)\}_{n=0}^{\infty}$.
(1) Test for Divergence fails.
(2) Test for Divergence gives divergence.
(3) Test for Divergence gives divergence.

6.2. **Worked-Out Solutions and Explanations.**

6.2.1. §*1.1.* First exercise:

(1) $\left|\dfrac{(-1)^{2n+1}}{8(n-5)}\right| = \dfrac{|(-1)^{2n+1}|}{|8||n-5|}$.

-1 raised to a power is always ± 1, so we can ignore the specifics of $2n+1$. The answer is $\dfrac{1}{8|n-5|}$ even if we know $n > 0$ because $n - 5$ could still be negative.

(2) $\left|\dfrac{-x^2}{4(n+10)^2}\right| = \dfrac{|-1||x^2|}{|4||(n+10)^2|}$.

Everything is already positive except for the -1. $\dfrac{x^2}{4(n+10)^2}$ is as simplified as possible.

(3) $\left|\dfrac{-3(n+2)}{n^3}\right| = \dfrac{|-3||n+2|}{|n^3|}$.
We can pull the absolute value inside the cube but we can't get rid of it since odd powers preserve sign. The answer with no conditions on n is $\dfrac{3|n+2|}{|n|^3}$; if we know $n > 0$ then both quantities inside absolute value signs must already be positive, so we can remove the absolute values entirely.

Second exercise:

(1) Multiply through by 2 and expand to get $-2 < x + 3 < 2$. Subtract 3 from all pieces. Then $-5 < x < -1$.

(2) Expand to $\frac{2x-1}{x+1} = \pm 3$. In the positive case, we have $2x - 1 = 3x + 3$, which gives $x = -4$. In the negative case, we have $2x - 1 = -3x - 3$, which gives $x = -2/5$.

(3) Multiplication by -1 flips the inequality and gives $\frac{1}{|2x-1|} < 2$. Before removing the absolute value, get it off the bottom of the fraction by cross-multiplying: $1/2 < |2x - 1|$. Now we expand to $1/2 < 2x - 1$ or $2x - 1 < -1/2$. The former gives $x > 3/4$ and the latter $x < 1/4$.

6.2.2. §*1.2.*

(1) $\sin^5 x \cos^2 x = \sin^4 x \cos^2 x \sin x = (\sin^2 x)^2 \cos^2 x \sin x$
$= (1 - \cos^2 x)^2 \cos^2 x \sin x$
$= (\cos^2 x - 2\cos^4 x + \cos^6 x)\sin x.$

(2) $\sin^4\theta = (\sin^2\theta)^2 = \left[\frac{1}{2}(1 - \cos(2\theta))\right]^2$
$= \frac{1}{4}\left[1 - \cos(2\theta) + \cos^2(2\theta)\right]$
$= \frac{1}{4}\left[1 - \cos(2\theta) + \frac{1}{2}(1 + \cos(4\theta))\right]$
$= \frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta).$

(3) $\tan^5 x \sec^3 x = \tan x(\tan^2 x)^2 \sec^3 x$
$= \tan x(\sec^2 x - 1)^2 \sec^3 x$
$= \tan x(\sec^7 x - 2\sec^5 x + \sec^3 x).$

(4) $\cos^3\theta \sin(2\theta) = \cos^3\theta[2\sin\theta\cos\theta]$
$= 2\sin\theta\cos^4\theta.$

6.2.3. §*1.3.* Note that in all of these, the triangle is not unique. What matters is the ratio of legs, not their exact length. Therefore we simply use the numbers given without manipulating them.

(1) $\cos(\sin^{-1}(2/3))$: sin is opp/hyp, so opp$=2$, hyp$=3$, and adj$=\sqrt{9-4}=\sqrt{5}$. Since cos is adj/hyp, this is $\sqrt{5}/3$.

(2) $\sec(\arctan(3))$: tan is opp/adj, so opp$=3$, adj$=1$, and hyp$=\sqrt{9+1}=\sqrt{10}$. Since sec is hyp/adj, this is $\sqrt{10}$.

(3) $\tan(\sec^{-1}(x/2))$: hyp$=x$, adj$=2$, and opp$=\sqrt{x^2-4}$. Hence this is $\frac{1}{2}\sqrt{x^2-4}$.

(4) $\sin(\arccos(x+1))$: adj$=x+1$, hyp$=1$, and opp$=\sqrt{1-(x+1)^2}=\sqrt{-(x^2+2x)}$. Hence this is $\sqrt{-(x^2+2x)}$.

6.2.4. §*2.1.*

(1) $\displaystyle\lim_{t\to 4}\frac{4-t}{2-\sqrt{t}}$.

- Method 1. View $4-t$ as $(2-\sqrt{t})(2+\sqrt{t})$. Cancel to get $\displaystyle\lim_{t\to 4}(2+\sqrt{t})=4$.
- Method 2. Multiply by $\dfrac{2+\sqrt{t}}{2+\sqrt{t}}$ to get $\displaystyle\lim_{t\to 4}\frac{(4-t)(2+\sqrt{t})}{4-t}$. Cancel and evaluate as above.

(2) $\displaystyle\lim_{h\to 0}\frac{(2+h)^2-4}{h}$.

Multiply out to get $\displaystyle\lim_{h\to 0}\frac{4h+h^2}{h}=\lim_{h\to 0}(4+h)=4$.

(3) $\displaystyle\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}$.

Multiply by $\dfrac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}$ and simplify to get

$$\lim_{x\to 0}\frac{x}{(\sqrt{3+x}+\sqrt{3})x}=\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}=\frac{1}{2\sqrt{3}}.$$

6.2.5. §*2.2.* Remember most of these simplifications are only legal because we can assume x is as large as we need.

(1) $\lim_{x\to\infty} \frac{\ln x}{\ln x^2}$

Recall that $\ln x^2 = 2\ln x$. Cancellation gives $1/2$.

(2) $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x}$

L'Hospital's rule gives $\lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{1/x} = \lim_{x\to\infty} \frac{1}{2}\sqrt{x}$, which is infinite.

(3) $\lim_{x\to\infty} \frac{3x-4}{\sqrt[3]{x^4+x^2-1}}$

The highest power here is effectively $x^{4/3}$, from the x^4 under the cube root. Multiplying through by $x^{-4/3}$ over itself will eliminate that exactly and turn everything else into negative powers of x:

$$\lim_{x\to\infty} \frac{3x-4}{\sqrt[3]{x^4+x^2-1}} \cdot \frac{x^{-4/3}}{x^{-4/3}} = \lim_{x\to\infty} \frac{\frac{3}{x^{1/3}} - \frac{4}{x^{4/3}}}{\sqrt[3]{1+\frac{1}{x^2}-\frac{1}{x^4}}}.$$

Now the limit laws apply. In brutal detail, this limit is equal to the following fraction of limits:

$$\frac{\lim\limits_{x\to\infty} \frac{3}{x^{1/3}} - \lim\limits_{x\to\infty} \frac{4}{x^{4/3}}}{\sqrt[3]{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \frac{1}{x^2} - \lim\limits_{x\to\infty} \frac{1}{x^4}}}.$$

Remember that technically the equality cannot be asserted until we've checked that all the component limits exist and this doesn't give us a denominator of 0. However, it all works out, giving $\frac{0}{1}$, for a limit of 0.

The limit within the text, $\lim\limits_{x\to\infty} e^x/x^3$, requires three rounds of L'Hospital's rule before you can evaluate it directly. The final limit is $\lim\limits_{x\to\infty} e^x/6$, which is clearly infinite.

6.2.6. §*3.1.* Find $L(x)$, dy and Δy for $f(x) = 3x^2 - 2$, $a = \frac{1}{2}$, and (first) $dx = \Delta x = 1$ and (second) $dx = \Delta x = -1$.

$f'(x) = 6x$; $f'(\frac{1}{2}) = 3$; $f(\frac{1}{2}) = -\frac{5}{4}$. Hence $L(x) = -\frac{5}{4} + 3(x - \frac{1}{2}) = 3x - \frac{11}{4}$.

For $dx = \Delta x = 1$, $\Delta y = f(\frac{3}{2}) - f(\frac{1}{2}) = 6$, and the differential $dy =$

$f'(\frac{1}{2})dx = 3$.
For $dx = \Delta x = -1$, $\Delta y = f(-\frac{1}{2}) - f(\frac{1}{2}) = 0$, and $dy = -3$.

Note that when we move the same distance from a but in the opposite direction, the actual change may vary in sign, magnitude, both, or neither, but the differential will keep its magnitude and flip its sign since it comes from movement along a straight line.

6.2.7. §*3.2.*

(1) $\displaystyle\int_{-1}^{0} \frac{1}{(2t+3)^{3.2}}\, dt.$

Let $u = 2t + 3$; then $du = 2dt$, -1 is converted to 1 and 0 is converted to 3.

$$\frac{1}{2}\int_1^3 u^{-3.2}\, du \;=\; \frac{-1}{4.4}\left(u^{-2.2}\right)\Big|_1^3 \;=\; \frac{1}{4.4}\left(1 - 3^{-2.2}\right).$$

(2) $\displaystyle\int \sec^3\theta \tan\theta\, d\theta \;=\; \int \sec^2\theta(\sec\theta\tan\theta)\, d\theta$

Now we can do a mental substitution of $u = \sec\theta$. The integral is $\frac{1}{3}\sec^3\theta + C$.

(3) $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)} \;=\; \int \frac{dx}{\sqrt{x}(1+(\sqrt{x})^2)}.$

This will be some version of arctan. If $u = \sqrt{x}$, so $du = \frac{dx}{2\sqrt{x}}$, we have

$$\int \frac{2\, du}{1+u^2} \;=\; 2\arctan u + C.$$

Put back into terms of x we get $2\arctan\sqrt{x} + C$.

(4) $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{1+\sin\theta}\, d\theta$

If $u = 1+\sin\theta$, $du = \cos\theta\, d\theta$. We don't need to do the whole substitution; this integral is

$$\ln(1+\sin\theta)|_0^{\pi/2} \;=\; \ln 2 - \ln 1 \;=\; \ln 2.$$

(5) $\displaystyle\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}} \;=\; \int \frac{\sec^2 t\, dt}{\sqrt{1+\tan t}}.$

If $u = 1+\tan t$, $du = \sec^2 t\, dt$; we can do this in our heads. The integral is $2\sqrt{1+\tan t} + C$.

(6) $\displaystyle\int x^3\sqrt{x^2+1}\,dx \;=\; \int x^2\sqrt{x^2+1}\,x\,dx.$

Let $u = x^2+1$, so $du = 2x\,dx$ and $x^2 = u-1$. We get
$$\frac{1}{2}\int (u-1)\sqrt{u}\,du \;=\; \frac{1}{2}\int (u^{3/2}-u^{1/2})\,du.$$

This is $\frac{1}{2}\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]+C$, which simplified and in terms of x is $\frac{1}{5}(x^2+1)^{5/2}-\frac{1}{3}(x^2+1)^{3/2}+C$.

(7) $\displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{\tan^{-1}x}}{1+x^2}\,dx.$

This is already fairly straightforward but can be made even more so by substituting $u = \tan^{-1}x$. The integral is
$$\frac{2}{3}(\tan^{-1}x)^{3/2}\Big|_1^{\sqrt{3}} \;=\; \frac{2}{3}\left[\left(\frac{\pi}{3}\right)^{3/2}-\left(\frac{\pi}{4}\right)^{3/2}\right],$$
which simplifies somewhat to $\dfrac{(8-3\sqrt{3})\pi^{3/2}}{36\sqrt{3}}$.

6.2.8. *§4.* First set of exercises:

(1) $0, 2, 4, 6, \ldots$ could be written $\{2n\}_{n=0}^{\infty}$. If we would like it to start with $n=1$ we need a function f such that $f(1)=0$ and $f(n+1)=f(n)+2$. Simply subtracting 2 from our current function will do it: $\{2n-2\}_{n=1}^{\infty}$.

(2) This sequence $9, 16, 25, 36, \ldots$ is the squares of integers 3 and greater. To start at 1 we could increase the value of n and then square: $\{(n+2)^2\}_{n=1}^{\infty}$.

The other starting point that makes sense is $n=3$: $\{n^2\}_{n=3}^{\infty}$.

(3) Let $f(n)=(n+1)^2$ and $g(n)=n^2+8n+16$ and suppose $\{f(n)\}_{n=k}^{\infty}$ and $\{g(n)\}_{n=\ell}^{\infty}$ give the same sequence.

Consider the first few terms of each sequence if we start at $n=0$. From f we get $1, 4, 9, 16, 25, \ldots$ (this looks familiar). From g we get $16, 25, 36, 49, \ldots$. We have to start three positions higher in f than in g in order to get the sequence elements to match up, so $k=\ell+3$.

(4) Suppose we modify the sequence in (1) so its terms are alternating in sign. We need to tack a -1 onto the appropriate terms of the sequence, the terms with odd n (using the original function $2n$). The way to do this is to give -1 an exponent which will be odd when n is odd and even when n is even. The easiest way to

do that is to use n itself: $\{(-1)^n(2n)\}_{n=0}^{\infty}$. If we had wanted the even-n terms to be negative, we could use the exponent $n+1$.

Musings on monotonicity and boundedness:

If we apply the definitions to functions, by saying *increasing* means $x_1 < x_2$ gives $f(x_1) < f(x_2)$ and so on, we can talk about the properties of the functions listed (let's restrict to, say, $x > 1$ to avoid some discontinuities). The limits are all at infinity.

- $f(x) = x$ is increasing and bounded from below only. It has limit infinity.
- $g(x) = 0$ is bounded, but not monotone. It has limit 0.
- $h(x) = \sin x$ is bounded, but not monotone. It has no limit.
- $r(x) = (\cos x)/x$ is bounded but not monotone. It has limit 1.
- $s(x) = 1/x$ is bounded and decreasing. It has limit 0.
- $t(x) = (x-1)/(x+1)$ is bounded and increasing. It has limit 1.

Any bounded, monotone sequence converges.

Second set of exercises:

(1) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$: the terms limit to 0 so this could converge, according to the Test for Divergence. It doesn't, however.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-3}{n^2+1}$: the terms limit to 2, so the Test for Divergence says this diverges.

(3) $\displaystyle\sum_{n=1}^{\infty} n^{1/5}$: the terms do not have a finite limit, so the Test for Divergence says this diverges.

Finding values of series:

I told you that the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is 1. You can actually prove this via *partial sums* s_i, what you get if you add the terms for n from 1 to i only. If $\lim_{i\to\infty} s_i$ exists, the series converges, and the value of the sum is the limit of the partial sums. You may have seen a similar trick to the one used here for proving $1+2+3+\ldots+n = \frac{1}{2}n(n+1)$; the idea is to use s_i and $\frac{1}{2}s_i$ to get a closed-form expression for s_i (instead of one involving $\ldots$). Then you can take a limit as usual.