10. Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region

$y = 1 + \sqrt{x}$ $\quad y = (3 + x)/3 = 1 + \frac{1}{3}x$

points of intersection

$1 + \sqrt{x} = 1 + 1/3\, x$

$1 - 1 + \sqrt{x} = 1 - 1 + 1/3\, x$

$\sqrt{x} = \frac{1}{3}x$

$3\sqrt{x} = 3 \cdot \frac{1}{3}x = x$

$(3\sqrt{x})^2 = x^2$

$9x = x^2$

$0 = x^2 - 9x = x(x-9)$

so the points of intersection occur when

$x = 0$, $x = 9$ thus the points of intersection

are $y = 1 + \sqrt{0} = 1$ giving $(0,1)$ and

$y = 1 + \sqrt{9} = 1 + 3 = 4$ giving $(9,4)$

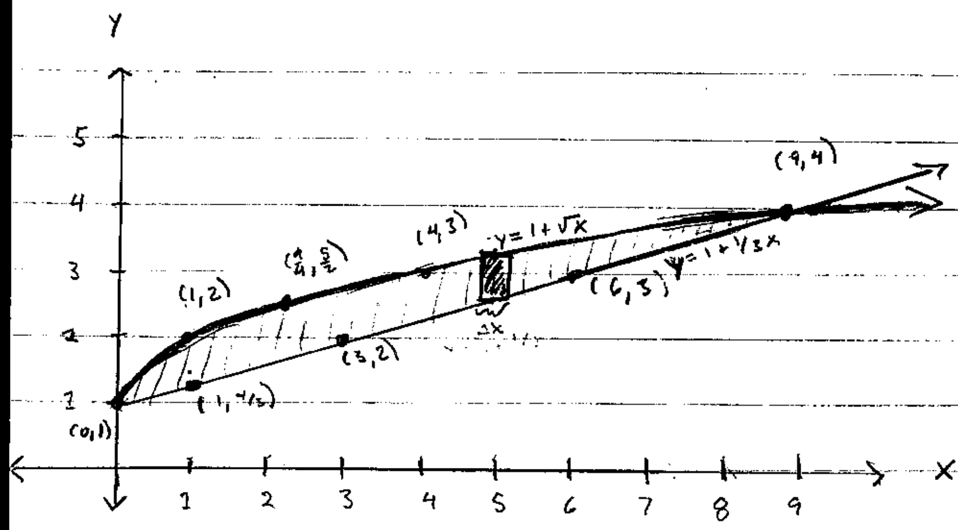
Graph: plot some points

$y = 1 + \sqrt{x}$

x	y
0	1
1	2
$\frac{9}{4}$	$\frac{5}{2}$
4	3
9	4

$y = 1 + \frac{1}{3}x$

x	y
0	1
1	$\frac{4}{3}$
3	2
6	3
9	4



Since both $y = 1 + 1/3\,x$ and $y = 1 + \sqrt{x}$ are fairly nice functions in x we will integrate with respect to x.

Now in our region from $x=0$ to $x=9$, $y = 1 + 1/3\,x$ is below $y = 1 + \sqrt{x}$, thus our integral will be

$$\int_0^9 (1+\sqrt{x}) - (1 + \tfrac{1}{3}x)\,dx = x + \tfrac{2}{3}x^{3/2} - x - \tfrac{1}{6}x^2 \Big|_0^9$$

$$= \tfrac{2}{3}(9)^{3/2} - \tfrac{1}{6}(9)^2 - \left(\tfrac{2}{3}(0)^{3/2} - \tfrac{1}{6}(0)^2\right)$$

$$= \tfrac{2}{3}\cdot 27 - \tfrac{1}{6}\cdot 81 = 2\cdot 9 - \tfrac{1}{2}\cdot 27 = 18 - \tfrac{27}{2} = \tfrac{36-27}{2} = \tfrac{9}{2}$$

18 Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region

$4x + y^2 = 12$ $\qquad$ $x = y$

solving for x, $x = 3 - \frac{1}{4}y^2$

points of intersection

$3 - \frac{1}{4}y^2 = y$

$\frac{1}{4}y^2 + y - 3 = 0$

$y^2 + 4y - 12 = 0$

$(y+6)(y-2) = 0$

thus the points of intersection occur when $y = 2$ and $y = -6$

the points of intersection are $x = y = 2$ giving $(2,2)$

and $x = y = -6$ giving $(-6,-6)$

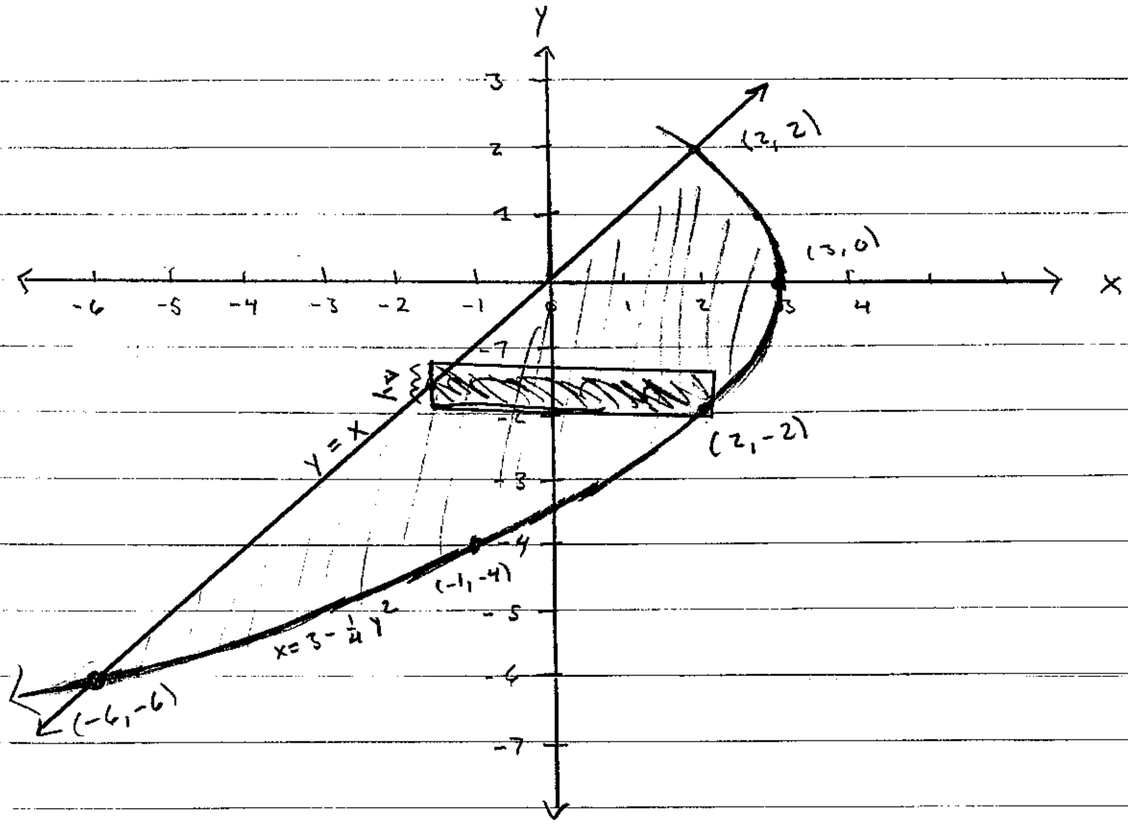
Graph: Plot some points

$x = 3 - \frac{1}{4}y^2$

x	y
-6	-6
-1	-4
2	-2
3	0
$\frac{11}{4}$	1
2	2

$x = y$

x	y
-6	-6
-4	-4
-2	-2
0	0
1	1
2	2



since both $x = 3 - \frac{1}{4}y^2$ and $y = x$ are fairly nice functions in y we will integrate with respect to y.

Now in our region from $y = -6$ to $y = 2$, $x = y$ is below $x = 3 - \frac{1}{4}y^2$

thus our integral will be

$$\int_{-6}^{2} 3 - \frac{1}{4}y^2 - y \, dy = 3y - \frac{1}{12}y^3 - \frac{1}{2}y^2 \Big|_{-6}^{2}$$

$$= 3(2) - \frac{1}{12}(2)^3 - \frac{1}{2}(2)^2 - \left(3(-6) - \frac{1}{12}(-6)^3 - \frac{1}{2}(-6)^2\right)$$

$$= 6 - \frac{2}{3} - 2 + 18 - 18 + 18 = \frac{64}{3}$$

24 Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region

$y = |x|$, $y = x^2 - 2$

points of intersection we need to solve $|x| = x^2 - 2$

there are two cases: $x < 0$ $x \geq 0$

if $x \geq 0$ then $|x| = x$ thus we need to solve

$x = x^2 - 2$

$0 = x^2 - x - 2 = (x-2)(x+1)$ since $x \geq 0$ $(x+1) \geq 1$ which implies

$(x-2)(x+1) = 0$ when $x - 2 = 0$, thus there is a point of intersection when $x = 2$

if $x < 0$ then $|x| = -x$ thus we need to solve

$-x = x^2 - 2$ or $0 = x^2 + x - 2 = (x+2)(x-1)$, now since $x < 0$

$x - 1 < -1$ which implies $(x+2)(x-1) = 0$ when $x + 2 = 0$, thus there is a point of intersection when $x = -2$.

Hence the points of intersection can be found by solving

$y = |2|$, $y = |-2|$ thus the points of intersection are $(-2,2)$, $(2,2)$

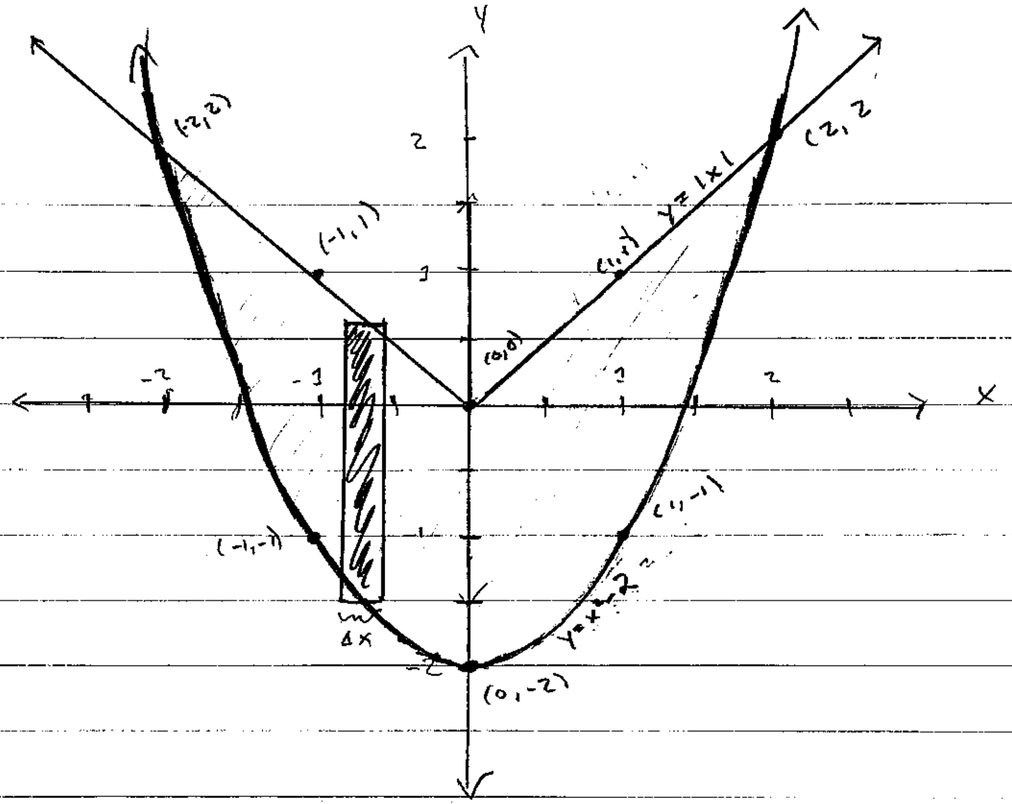
Graph plot some points

$y = |x|$

x	y
-2	2
-1	1
0	0
1	1
2	2

$y = x^2 - 2$

x	y
-2	2
-1	-1
0	-2
1	-1
2	2



Since both $y=x^2-2$ and $y=|x|$ are fairly nice functions in x we will integrate with respect to x.

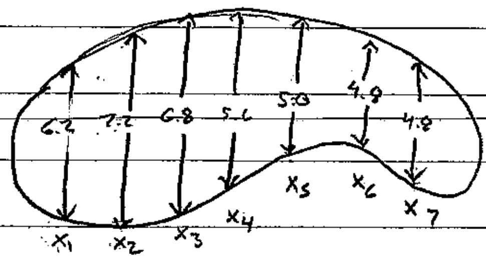
Now in our region from $x=-2$ to $x=2$, $y=x^2-2$ is below $y=|x|$ thus our integral is

$$\int_{-2}^{2} |x|-(x^2-2)\,dx = \int_{-2}^{0} -x-x^2+2 \quad + \int_{0}^{2} x-x^2+2$$

$$= \left[-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3+2x\right]\Big|_{-2}^{0} + \left[\tfrac{1}{2}x^2-\tfrac{1}{3}x^3+2x\right]\Big|_{0}^{2}$$

$$= \left[-\tfrac{1}{2}(0)^2-\tfrac{1}{3}(0)^3+2(0)-\left(-\tfrac{1}{2}(-2)^2-\tfrac{1}{3}(-2)^3+2(-2)\right)\right]$$
$$+\left[\tfrac{1}{2}(2)^2-\tfrac{1}{3}(2)^3+2(2)-\left(\tfrac{1}{2}(0)^2-\tfrac{1}{3}(0)^3+2(0)\right)\right]$$
$$= 2-\tfrac{8}{3}+4+2-\tfrac{8}{3}+4 = \tfrac{20}{3}$$

40. The widths (in meters) of a kidney shaped swimming pool were measured at 2 meter intervals as indicated in the figure. Use the midpoint rule to estimate the area of the pool.



We use the midpoint rule with $n=4$ intervals so that $\Delta x = \frac{2 \cdot 8}{4} = 4$

the midpoints are $\bar{x}_1 = 6.2$, $\bar{x}_2 = 6.8$, $\bar{x}_3 = 5.0$, $\bar{x}_4 = 4.8$

we estimate the area of the pool as follows

$$A \approx \Delta x\,[6.2 + 6.8 + 5.0 + 4.8] = 4[22.8] = 91.2$$

so the area of the pool is approximately 91.2 square meters

44 Find the area bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1,1)$ and the x-axis.

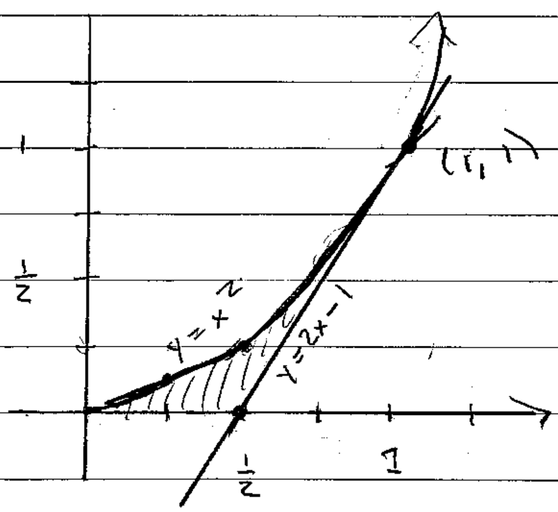
First we must find the tangent line to the parabola at $(1,1)$

$y' = 2x$ so the slope of the tangent line at the point $(1,1)$ is $y'(1) = 2(1) = 2$

Next we know the point $(1,1)$ is on this line

thus using point slope formula we get the equation of the tangent line is $y = 2x - 1$

Next we will sketch our region



Thus we will integrate

$$\int_0^{\frac{1}{2}} x^2 \, dx + \int_{1/2}^{1} x^2 - 2x + 1 \, dx = \frac{1}{3} x^3 \Big|_0^{1/2} + \left[\frac{1}{3} x^3 - x^2 + x \right]_{1/2}^{1}$$

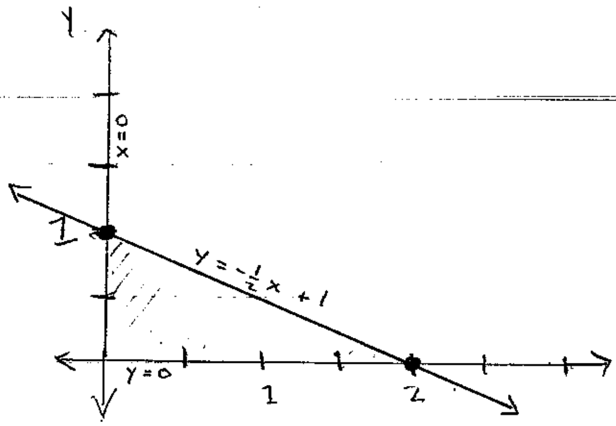
$$= \frac{1}{3} \cdot \frac{1}{8} + \frac{1}{3} - 1 + 1 - \left[\frac{1}{3} \cdot \frac{1}{8} - \frac{1}{4} + \frac{1}{2} \right] = \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{4 + 3 - 6}{12} = \frac{1}{12}$$

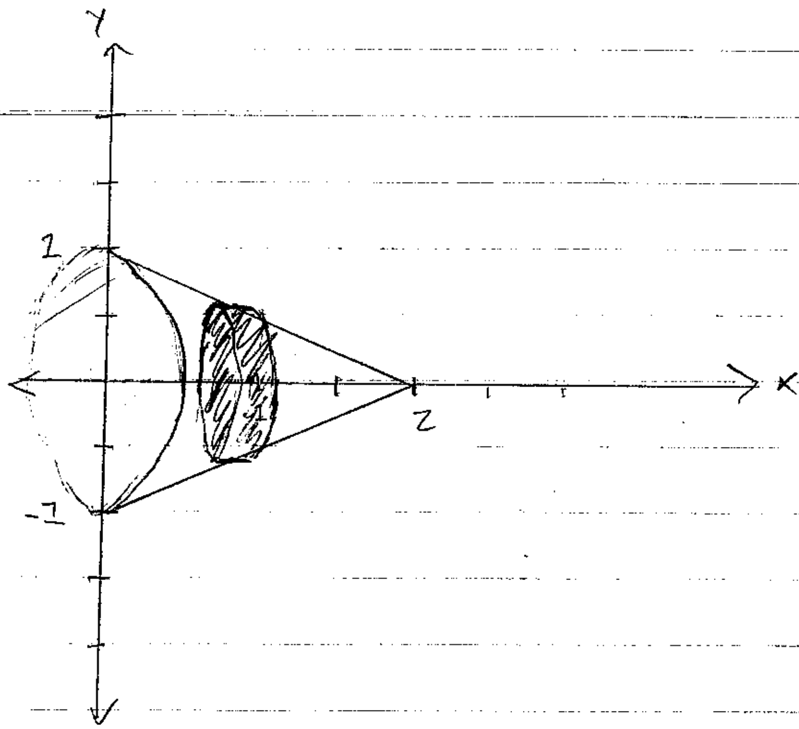
the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

$x+2y=2$, $x=0$ $y=0$ about the x-axis

so solve for y

$y=-\frac{1}{2}x+1$





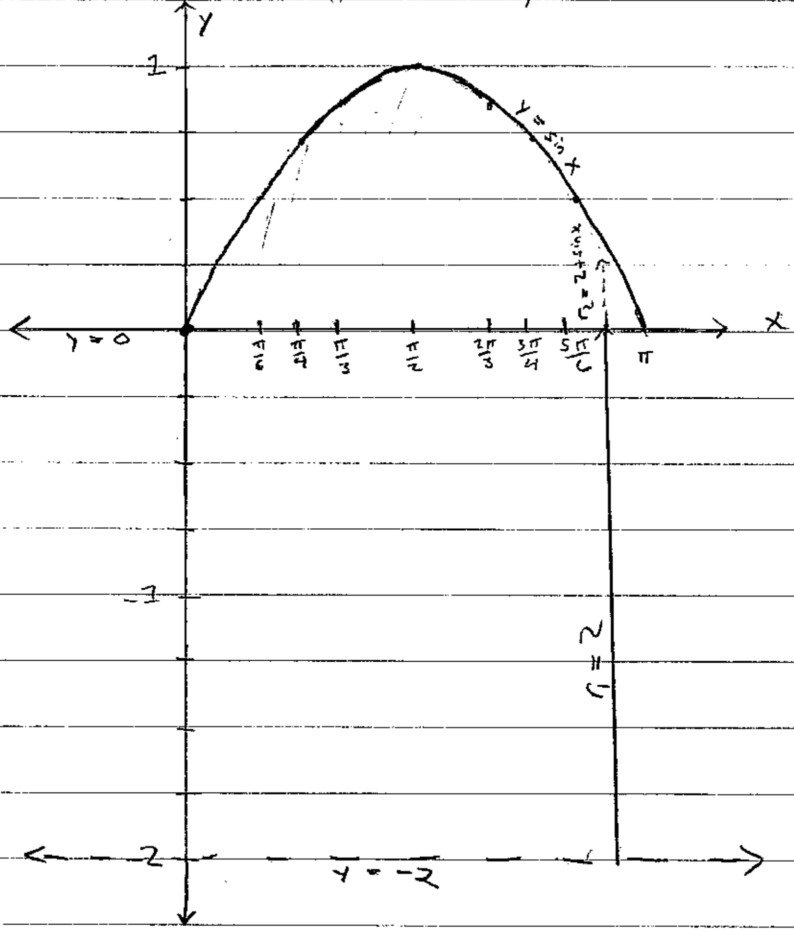
$$V=\int_0^2 A(x)\,dx=\int_0^2 \pi r^2 dx=\int_0^2 \pi\left(-\tfrac{1}{2}x+1\right)^2 dx$$

$$=\pi\int_0^2 \tfrac{1}{4}x^2-x+1\,dx=\pi\left(\left[\tfrac{1}{12}x^3-\tfrac{x^2}{2}+x\right]\Big|_0^2\right)$$

$$=\pi\left(\tfrac{1}{12}(2)^3-\tfrac{1}{2}(2)^2+2\right)=\pi\left(\tfrac{8}{12}-2+2\right)=\tfrac{8\pi}{12}=\tfrac{2\pi}{3}$$

34 Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves and the specified line

$y = 0 \quad y = \sin x, \quad 0 \le x \le \pi \quad$ about $\; y = -2$



$$V = \int_0^{\pi} \pi \left[r_2^2 - r_1^2 \right] = \pi \int_0^{\pi} (2 + \sin x)^2 - 2^2 \, dx = \pi \int_0^{\pi} 4 + 2 \sin x + \sin^2 x - 4 \, dx$$

$$= \pi \int_0^{\pi} 2 \sin x + \sin^2 x$$

36 Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

$2x+3y=6$, $(y-1)^2=4-x$; about $x=-5$

solve for x

$x=-\frac{3}{2}y+3$, $x=4-(y-1)^2$

points of intersection

$-\frac{3}{2}y+3 = 4-(y-1)^2$

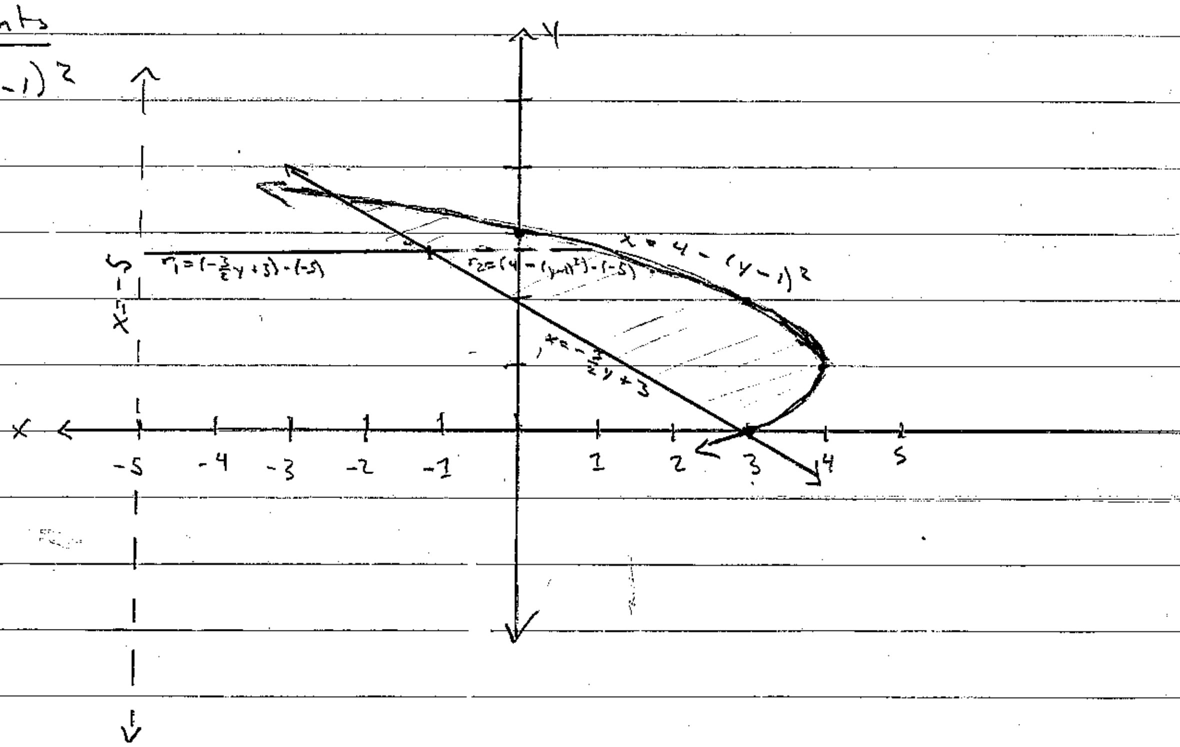
$0 = 1+\frac{3}{2}y - y^2 + 2y - 1 = \frac{7}{2}y - y^2 = y(\frac{7}{2}-y)$

so the points of intersection occur when $y=0$ and $y=7/2$

plot points

$x = 4-(y-1)^2$

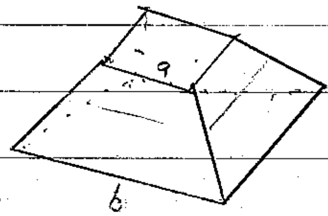
x	y
3	0
4	1
$\frac{15}{4}$	$\frac{3}{2}$
3	2
$\frac{7}{4}$	$\frac{5}{2}$
0	3
$-\frac{9}{4}$	$\frac{7}{2}$



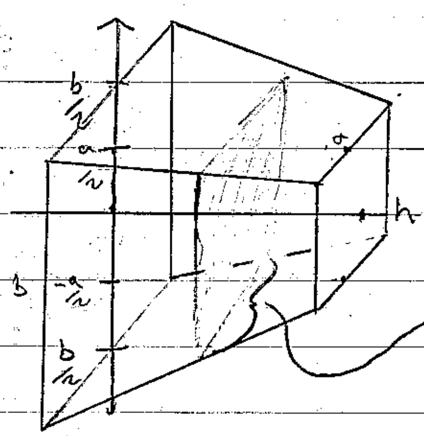
$$V = \int_0^{7/2} A(y)\,dy = \int_0^{7/2} \pi(r_2^2 - r_1^2)\,dy = \pi\int_0^{7/2} (9-(y-1)^2)^2 - (-\tfrac{3}{2}y+8)^2\,dy$$

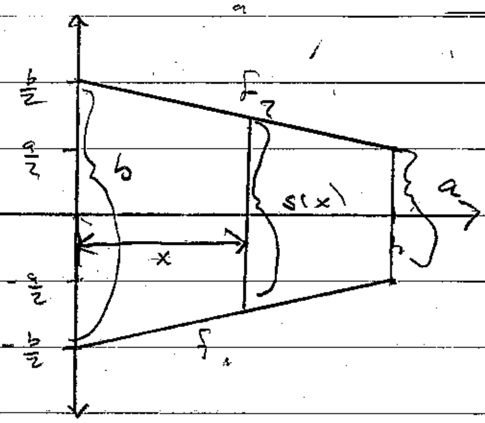
$$= \pi\int_0^{7/2} 81 - 18(y-1)^2 + (y-1)^4 - \tfrac{9}{4}y^2 + 24y - 64\,dy$$

50 Find the volume of the described solid
A frustrum of a pyramid with square base of side b, square top of side a, and height h.



what happens if $a = b$? What happens if $a = 0$





so if we solve the equations for the lines f_1, f_2 we can find s in terms of x

the slope of f_2 is, $m_2 = \dfrac{\frac{a}{2} - \frac{b}{2}}{h} = \dfrac{a-b}{2h}$

the slope of f_1 is $m_1 = \dfrac{-\frac{a}{2} - (-\frac{b}{2})}{h} = \dfrac{b-a}{2h}$

the y intersepts are $\frac{b}{2}$ and $-\frac{b}{2}$ respectively

which implies: $f_2 = \frac{a-b}{2h}x + \frac{b}{2}$ and $f_1 = \frac{b-a}{2h}x - \frac{b}{2}$

Now $= f_2(x) - f_1(x) = \frac{a-b}{2h}x + \frac{b}{2} - \left(\frac{b-a}{2h}x - \frac{b}{2}\right)$

$= \frac{a-b}{h}x + b$

Hence $A(x) = s(x)^2 = \left(\frac{a-b}{h}x + b\right)^2$

Thus $V = \displaystyle\int_0^h \left(\frac{a-b}{h}x + b\right)^2 dx = \int_0^h \left(\frac{a-b}{h}\right)^2 x^2 + \frac{2b(a-b)}{h}x + b^2\, dx =$

$$= \left(\frac{a-b}{h}\right)^2 \frac{x^3}{3} + \frac{b(a-b)}{h} x^2 + b^2 x \Big|_0^h$$

$$= \left(\frac{a-b}{h}\right)^2 \frac{h^3}{3} + \frac{b(a-b)}{h} h^2 + b^2 h$$

$$= \frac{1}{3}(a-b)^2 h + b(a-b)h + b^2 h = \frac{1}{3}(a^2 - 2ab + b^2)h + abh - b^2 h + b^2 h =$$

$$= \frac{1}{3}a^2 h - \frac{2}{3}abh + \frac{1}{3}b^2 h + abh = \frac{1}{3}a^2 h + \frac{1}{3}abh + \frac{1}{3}b^2 h$$

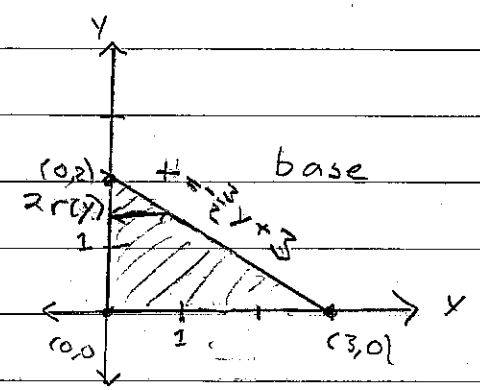
$$= \frac{h}{3}(a^2 + ab + b^2)$$

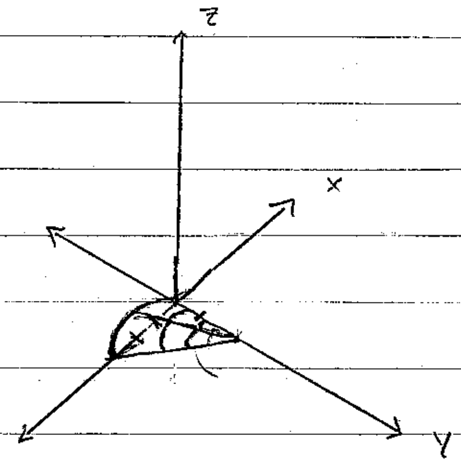
if $a = 0$ then $V = \frac{1}{3} h b^2$ (volume of cone)

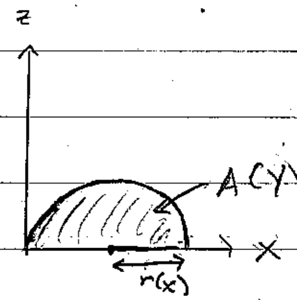
if $a = b$ then $V = \frac{h}{3}(b^2 + b^2 + b^2) = h b^2$ (volume of box)

58 Find the volume of the described solid.

The base of S is the triangular region with vertices $(0,0)$, $(3,0)$ and $(0,2)$. Cross-sections perpendicular to the y axis are semi-circles.







$$r(y) = \left(-\tfrac{3}{2}y + 3\right)/2 = -\tfrac{3}{4}y + \tfrac{3}{2}$$

Thus $A(y) = \frac{1}{2}\pi\, r(y)^2 = \frac{1}{2}\pi\left(-\frac{3}{4}y + \frac{3}{2}\right)^2$

Hence

$$V(S) = \int_0^2 A(y)\,dy = \int_0^2 \frac{1}{2}\pi\left(-\frac{3}{4}y + \frac{3}{2}\right)^2 dy$$

Let $u = -\frac{3}{4}y + \frac{3}{2}$ then $du = -\frac{3}{4}dy$ giving

$$\frac{1}{2}\pi\int_0^2 \left(-\frac{3}{4}y + \frac{3}{2}\right)^2 dy = -\frac{4}{2}\pi\int_{3/2}^0 u^2\,du = \frac{2}{3}\pi\int_0^{3/2} u^2\,du$$

$$= \frac{2}{3}\pi \cdot \frac{1}{3}u^3\Big|_0^{3/2} = \frac{2}{3}\pi \cdot \frac{1}{3}\cdot\frac{27}{8} = \frac{3}{4}\pi$$