Homework due Oct. 6th

1. $1+0.4+0.16+0.064+\ldots = \sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^{n-1}$

$= \frac{1}{1-2/5}$ since this is a geometric series with $r=\frac{2}{5}<1$

$= \frac{1}{(3/5)} = \frac{5}{3}$

2. $\sum_{n=1}^{\infty}\frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty} e\cdot\frac{e^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} e\cdot\left(\frac{e}{3}\right)^{n-1}$

$= \frac{e}{1-\frac{e}{3}}$ as a geometric series, with $\left|\frac{e}{3}\right|<1$ because $0<e<3$.

$= \frac{e}{\left(\frac{3-e}{3}\right)} = \frac{3e}{3-e}$

3. $0.\overline{73} = \frac{73}{10^2}+\frac{73}{10^4}+\frac{73}{10^6}+\ldots = \sum_{n=1}^{\infty} 73\cdot\frac{1}{10^{2n}}$

$= \sum_{n=1}^{\infty}\frac{73}{10^2}\cdot\left(\frac{1}{10^2}\right)^{n-1}$

$= \frac{73/10^2}{1-\frac{1}{10^2}} = \frac{73}{10^2}\cdot\frac{10^2}{99} = \frac{73}{99}$

4. We know that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.
Since, for each n, $\frac{3}{n}>\frac{1}{n}>0$, $\sum_{n=1}^{\infty}\frac{3}{n}$ diverges.

5. We have the partial sums

$$S_n = \sum_{j=2}^{n}\frac{2}{j^2-1} = \sum_{j=2}^{n}\frac{2}{(j-1)(j+1)} = \sum_{j=2}^{n}\left(\frac{1}{j-1}-\frac{1}{j+1}\right)$$

by partial fractions.

i.e. $S_n = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \tfrac{1}{5}\right) + \left(\cancel{\tfrac{1}{4}} - \tfrac{1}{6}\right) + \dots + \left(\tfrac{1}{n-1} - \tfrac{1}{n+1}\right)$

Notice this is a telescoping series, with

$$S_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

Thus $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = \frac{3}{2}$

6. $\displaystyle\lim_{n\to\infty} \ln\left(\frac{n}{2n+5}\right) = \lim_{n\to\infty} \ln\left(\frac{1}{2+5/n}\right) = \ln\left(\frac{1}{2}\right) \neq 0$

so $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5}\right)$ diverges.

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x+3}{2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{x+3}{2}\right)^{n-1}$

This is a geometric series, with $r = \frac{x+3}{2}$, so it converges if and only if $|r| < 1$

i.e. $-1 < \frac{x+3}{2} < 1$

$-2 < x+3 < 2$

$-5 < x < -1$

For these values of x only, the sum is convergent,

and $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n} = \frac{1}{1 - \frac{x+3}{2}} = \frac{2}{(2-(x+3))} = \frac{-2}{x+1}$