Homework Due October 10, 2004

①-④ Find the radius of convergence and interval of convergence of the series.

① $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ $a_n = \frac{(-1)^n x^n}{n+1}$ We use the Ratio Test to see when the series converges. $\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+2}}{\frac{x^n}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right| =$

$\lim_{n\to\infty} \frac{|x|}{1+\frac{1}{n+1}} = |x|$. So by the Ratio Test, we know the series converges when $|x|<1$, so $R=1$. Now to find I. When $x=-1$, the series is the harmonic series, and so it diverges. When $x=1$, the series is the Alternating Harmonic series, and so it converges by the Alternating Series test. So we have $I=(-1,1]$.

② $\sum_{n=1}^{\infty} \frac{x^3}{n3^n}$ $a_n = \frac{x^n}{n3^n}$ We shall follow the same method as above.

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{xn}{(n+1)3} \right| = \frac{|x|}{3} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x|}{3}$

So the series converges when $\frac{|x|}{3}<1$, $|x|<3$. So $R=3$.
When $x=-3$, the series is the alternating harmonic series, which we know converges by the Alternating Series Test. When $x=3$, the series is the harmonic series which we know diverges. So $I=[-3,3)$

(3) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ We proceed as before.

$a_n = \frac{(-1)^n x^{2n}}{(2n)!}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{|x|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^{2n}}\right| = \lim_{n\to\infty} \frac{|x|^2}{(2n+1)(2n+2)} = 0$

So, by the Ratio test, this series converges for all real numbers. So $R=\infty$ & $I=(-\infty,\infty)$.

(4) $\sum_{n=0}^{\infty} n^3(x-5)^n$ We proceed as before.

$a_n = n^3(x-5)^n$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^3 |x-5|$

$= |x-5|$. By the Ratio Test, the series converges when $|x-5|<1 \Leftrightarrow -1<x-5<1 \Leftrightarrow 4<x<6$ So $R=1$.

When $x=4$, the series is $\sum_{n=0}^{\infty} (-1)^n n^3$ which diverges.

When $x=6$, the series is $\sum_{n=0}^{\infty} n^3$ which also diverges. So $I=(4,6)$

(5) & (6) Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x=-4$ and diverges when $x=6$. What can be said about the convergence or divergence of the following series?

(5) $\sum_{n=0}^{\infty} c_n 8^n$ Here $x=8$. We know that the series above diverges at least when $x<-6$ or $x\geq 6$. Since $8>6$, this series is <u>divergent</u>.

(6) $\sum_{n=0}^{\infty} c_n (-3)^n$ Here $x=-3$. We know the above series converges at least when $-4\leq x<4$. Since $-3\geq -4$ & $-3<4$, this series is <u>convergent</u>.

7 & 8 Find a power series representation for the function & determine the interval of convergence

7 $f(x) = \dfrac{3}{1-x^4} = 3\left(\dfrac{1}{1-x^4}\right) = 3(1+x^4+x^8+x^{12}+x^{16}+\dots) = 3\sum_{n=0}^{\infty}(x^4)^n$

$= \sum_{n=0}^{\infty} 3x^{4n}$ This converges $\Leftrightarrow$ $\sum_{n=0}^{\infty}(x^4)^n$ converges. So $\sum_{n=0}^{\infty} 3x^{4n}$

converges when $|x^4|<1 \Leftrightarrow |x|<1$, so $R=1$ & $I=(-1,1)$

8 $f(x) = \dfrac{x}{4x+1} = x\left(\dfrac{1}{1-(-4x)}\right) = x\sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-1)^n 4^n x^{n+1}$

This converges when $|-4x|<1 \Leftrightarrow |x|<\frac{1}{4}$ $R=\frac{1}{4}$ $I=\left(-\frac{1}{4},\frac{1}{4}\right)$

9 (a) Use differentiation to find a power series representation for $f(x) = \dfrac{1}{(1+x)^2}$. What is the radius of convergence?

Note that $\dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = \dfrac{1}{(1+x)^2}$

$\dfrac{-1}{1+x} = \dfrac{-1}{1-(-x)} = -\sum_{n=0}^{\infty}(-x)^n = -\sum_{n=0}^{\infty}(-1)^n(x)^n$

So $f(x) = \dfrac{1}{(1+x)^2} = -\dfrac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} = \sum_{n=0}^{\infty}(-1)^{n+2}(n+1)x^n$

$= \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ Here, $R=1$, since $|-x|<1 \Leftrightarrow |x|<1$.

(b) Use part (a) to find a power series for $f(x) = \frac{1}{(1+x)^3}$

Note that $\frac{d}{dx} \frac{1}{(1+x)^2} = -2\left(\frac{1}{(1+x)^3}\right)$

So $f(x) = \frac{1}{(1+x)^3} = \frac{-1}{2} \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} (n+1) n \, x^{n-1}$

$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n$ Again, $R=1$

(c) Use part (b) to find a power series for $f(x) = \frac{x^2}{(1+x)^3}$

$f(x) = x^2 \cdot \frac{1}{(1+x)^3} = x^2 \cdot \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^{n+2}$

We can rewrite this as $\frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n)(n-1) x^n$. Note the change in the initial value of n.

(10) Find a power series representation for $f(x) = \ln(1+x)$. What is the radius of convergence

$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n$ (Here $R=1$)

So, $f(x) = \ln(1+x) = \int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} + C$ $f(0) = \ln(1) = 0$, so $C=0$. & $R=1$.

11 Use 10 to find a power series for $f(x) = x\ln(1+x)$.

$$f(x) = x\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n+1}}{n} = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n-1} \qquad (R=1)$$