Math 8
Due Oct 15

1) What can you say about the series Σa_n in each of the following cases?

(a) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 8$

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 8 > 1$ so by part (ii) of the ratio test the series is divergent

(b) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8$

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8 < 1$ so by part (i) of the ratio test the series is absolutely convergent which implies the series is convergent

(c) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

by part (iii) of the ratio test we can say nothing about the convergence properties of the series, we would have to apply another test

2) Determine whether the series is absolutely convergent, conditionally convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Applying the ratio test we see $a_n = \frac{n^2}{2^n}$ and $a_{n+1} = \frac{(n+1)^2}{2^{n+1}}$ which implies

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{2^n(n+1)^2}{2^{n+1}\ n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2^n(n^2+2n+1)}{2\cdot 2^n(n^2)}\right| = \lim_{n\to\infty}\left|\frac{n^2+2n+1}{2n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^2(1+\frac{2}{n}+\frac{1}{n^2})}{2n^2}\right| = \lim_{n\to\infty}\left|\frac{1+\frac{2}{n}+\frac{1}{n^2}}{2}\right|$$

$$= \frac{1}{2} < 1$$

which implies that the series is absolutely convergent

3) Determine whether the series is absolutely convergent conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$$

Applying the ratio test we get $a_n = \dfrac{(-1)^{n+1} n^2 2^n}{n!}$

and $a_{n+1} = \dfrac{(-1)^{n+2}(n+1)^2 2^{n+1}}{(n+1)!}$

so $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(n+1)^2 2^{n+1}}{(n+1)!}}{\frac{(-1)^{n+1} n^2 2^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2} n! (n+1)^2 2^{n+1}}{(-1)^{n+1}(n+1)! n^2 2^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-1\, \cancel{n!}\, (n+1)^{\cancel{2}}\, 2\cdot\cancel{2^n}}{\cancel{(n+1)}\,\cancel{n!}\; n^2\, \cancel{2^n}}\right| = \lim_{n\to\infty}\left|-\left(\frac{2(n+1)}{n^2}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n+2}{n^2}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n^2}\left(\frac{2}{n}+\frac{2}{n^2}\right)}{\cancel{n^2}}\right| = \lim_{n\to\infty}\left|\frac{2}{n}+\frac{2}{n^2}\right| = 0$$

Thus by the ratio test it is absolutely convergent

4) Determine whether the series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

using the ratio test $a_n = \frac{n!}{n^n}$, $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

so $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \right| = \lim_{n\to\infty} \left| \frac{n^n (n+1)!}{(n+1)^{n+1}\, n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^n (n+1)\, n!}{(n+1)^n (n+1)\, n!} \right| = \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^n} \right| = \lim_{n\to\infty} \left| \frac{1}{\frac{(n+1)^n}{n^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{\left(\frac{n+1}{n}\right)^n} \right| = \lim_{n\to\infty} \left| \frac{1}{\left(1+\frac{1}{n}\right)^n} \right| \overset{*}{=} \frac{1}{e} < 1$$

Thus by the ratio test the series is absolutely convergent.

* Please see example 5 in this section and equation 7.4.9

note on problem 4

if $f(x) = \ln(x)$ then $f'(x) = 1/x$. Thus $f'(1) = 1$

we will use this to show the $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

from the definition of the derivative we have

$$1 = f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \ln(1+h) = \lim_{h\to 0} \ln(1+h)^{1/h}$$

Thus $\lim_{h\to 0} \ln(1+h)^{1/h} = 1$

by the continuity of e^x we get

$$e = e^1 = \lim_{h\to 0} e^{\ln(1+h)^{1/h}} = \lim_{h\to 0} (1+h)^{1/h}$$

if we let $n = \frac{1}{h}$ then as $h \to 0$ $n \to \infty$ thus

$$e = \lim_{h\to 0} (1+h)^{1/h} = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

5) Determine whether the series is absolutely convergent, conditionally convergent or divergent.

$$\frac{2}{5} + \frac{2 \cdot 6}{5 \cdot 8} + \frac{2 \cdot 6 \cdot 10}{5 \cdot 8 \cdot 11} + \frac{2 \cdot 6 \cdot 10 \cdot 14}{5 \cdot 8 \cdot 11 \cdot 14} + \cdots$$

It is easy to see the pattern here but it is hard to find the general expression for each term a_n of this series. However using the ratio test we only need to find an expression for $\frac{a_{n+1}}{a_n}$ which in this case is easier to do.

Now $$\frac{a_2}{a_1} = \frac{\frac{2 \cdot 6}{5 \cdot 8}}{\frac{2}{5}} = \frac{2 \cdot 5 \cdot 6}{2 \cdot 5 \cdot 8} = \frac{6}{8}$$

$$\frac{a_3}{a_2} = \frac{\frac{2 \cdot 6 \cdot 10}{5 \cdot 8 \cdot 11}}{\frac{2 \cdot 6}{5 \cdot 8}} = \frac{2 \cdot 6 \cdot 5 \cdot 8 \cdot 10}{2 \cdot 6 \cdot 5 \cdot 8 \cdot 11} = \frac{10}{11}$$

similarly $\frac{a_4}{a_3} = \frac{14}{14}$ $\qquad \frac{a_5}{a_4} = \frac{18}{17}$ $\qquad \frac{a_6}{a_5} = \frac{22}{20}$

notice that the differences in the numerator $(14-10) = 4 = (10-6)$

so the numerator will be $4n+b$ for some constant b

similarly the differences in the denominator are $(11-8) = 3 = (14-11)$

so the denominator will be $3n+c$ for some constant c.

to find b, c notice for $n=1$ $\quad \frac{a_{n+1}}{a_n} = \frac{6}{8} = \frac{4n+b}{3n+c}$

so $4(1) + b = 6$ giving $b = 2$ and $3(1) + c = 8$ giving $c = 5$

thus $\frac{a_{n+1}}{a_n} = \frac{4n+2}{3n+5}$

hence $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{4n+2}{3n+5} \right| = \lim_{n\to\infty} \left| \frac{n\left(4+\frac{2}{n}\right)}{n\left(3+\frac{5}{n}\right)} \right|$

$= \lim_{n\to\infty} \left| \frac{4+\frac{2}{n}}{3+\frac{5}{n}} \right| = \frac{4}{3} > 1$ thus

by the ratio test this series diverges ∎

6) For which of the following series is the Ratio Test inconclusive.

Before we look at the series, notice that the ratio test is inconclusive if and only if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ $\quad a_n = \frac{1}{n^3} \quad a_{n+1} = \frac{1}{(n+1)^3}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}}\right| = \lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\right| = \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^3\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n}{n(1+\frac{1}{n})}\right)^3\right| = \lim_{n\to\infty}\left|\frac{1}{(1+\frac{1}{n})^3}\right| = 1$$

inconclusive
(of course the p-series test shows it converges)

(b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ $\quad a_n = \frac{n}{2^n} \quad a_{n+1} = \frac{n+1}{2^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{2^n(n+1)}{2^{n+1}(n)}\right| = \lim_{n\to\infty}\left|\frac{1}{2}\left(\frac{n+1}{n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{2}\left(1+\frac{1}{n}\right)\right| = \frac{1}{2} < 1$$

The ratio test gives that this series converges absolutely

(c) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$ $\quad a_n = \frac{(-3)^{n-1}}{\sqrt{n}} \quad a_{n+1} = \frac{(-3)^n}{\sqrt{n+1}}$

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^n}{\sqrt{n+1}}}{\frac{(-3)^{n-1}}{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{\sqrt{n}\ (-3)^n}{\sqrt{n+1}\ (-3)^{n-1}} \right| = \lim_{n\to\infty} \left| -3 \sqrt{\frac{n}{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| 3 \sqrt{\frac{n}{n(1+\frac{1}{n})}} \right| = \lim_{n\to\infty} \left| 3 \sqrt{\frac{1}{1+\frac{1}{n}}} \right| = 3 > 1$$

Thus by the ratio test the series diverges.

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$ $\quad a_n = \frac{\sqrt{n}}{1+n^2} \quad a_{n+1} = \frac{\sqrt{1+n}}{1+(n+1)^2}$

$$\lim_{n\to\infty} \left| \frac{\frac{\sqrt{1+n}}{1+(n+1)^2}}{\frac{\sqrt{n}}{1+n^2}} \right| = \lim_{n\to\infty} \left| \frac{(1+n^2)\sqrt{1+n}}{\sqrt{n}\,(1+(n+1)^2)} \right| = \lim_{n\to\infty} \left| \frac{\sqrt{1+n} + n^2\sqrt{1+n}}{\sqrt{n}\,(2+2n+n^2)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\sqrt{1+n} + \sqrt{n^4+n^5}}{2\sqrt{n} + 2n\sqrt{n} + n^2\sqrt{n}} \right| = \lim_{n\to\infty} \left| \frac{\sqrt{n^5}\left(\sqrt{\frac{1}{n^5}+\frac{1}{n^4}} + \sqrt{\frac{1}{n}+1}\right)}{\sqrt{n^5}\left(2\left(\sqrt{\frac{1}{n^4}}\right) + 2\sqrt{\frac{1}{n^2}} + 1\right)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\sqrt{\frac{1}{n^5}+\frac{1}{n^4}} + \sqrt{\frac{1}{n}+1}}{\frac{2}{n^2} + \frac{2}{n} + 1} \right| = 1$$

The ratio test is inconclusive

(using the comparison test, we know $n^2+1 > n^2$ thus $\frac{1}{n^2+1} < \frac{1}{n^2}$ hence $\frac{\sqrt{n}}{n^2+1} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ for all n, $\sum \frac{1}{n^{3/2}}$ is a p-series and $3/2 > 1$ thus it converges implying $\sum \frac{\sqrt{n}}{n^2+1}$ converges)

Hence the ratio test is inconclusive for (a) and (d)