# *LECTURE OUTLINE*

## *Velocity, acceleration, and curvature*

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Math 8

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## *Goals*

Velocity

Acceleration

Curvature

# *Space Curve review*

Recall our Helix:

$$\vec{v}(t) =< \cos(t), \sin(t), t > .$$

Sketch the helix. Find the velocity at each time of particle traveling along a helix. Watch is the direction of this velocity, this is usually denoted $\hat{T}(t)$ and called the *unit tangent vector*. Find $\frac{d}{dt}\hat{T}(t)$'s direction vector for our helix, and call it $\hat{N}(t)$ the *curve's normal vector*.

Recall from last time: $\hat{T} \cdot \hat{N} = 0$

Let $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$, this is usually called the curve's *binormal vector*. Express $\frac{d}{dt}\hat{N}(t)$ as $-a(t)\hat{T} + b(t)\hat{B}$ for our helix. Why is this always true?

Compute $\hat{B}$ and $\frac{d}{dt}\hat{B}$ for our helix. Why must $\frac{d}{dt}\hat{B}$ always be a multiple of $\hat{N}$?

# *Differentiation Rules*

1. $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \frac{d\vec{u}}{dt}(t) + \frac{d\vec{v}}{dt}(t)$
2. $\frac{d}{dt}[c\vec{u}(t)] = c\frac{d\vec{u}}{dt}(t)$
3. $\frac{d}{dt}[f(t)\vec{u}(t)] = f(t)\frac{d\vec{u}}{dt}(t) + \frac{df}{dt}(t)\vec{u}$
4. $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \frac{d\vec{u}}{dt}(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \frac{d\vec{v}}{dt}(t)$
5. $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \frac{d\vec{u}}{dt}(t) \times \vec{v}(t) + \vec{u}(t) \times \frac{d\vec{v}}{dt}(t)$
6. $\frac{d}{dt}[\vec{u}(f(t))] = \frac{d\vec{u}}{dt}(t)\frac{df}{dt}(t)$

# *Acceleration*

The *acceleration* of our particle $\vec{r}(t)$ is

$$\frac{d^2}{dt^2}\vec{r}(t).$$

For $t \geq a$

$$\frac{d}{dt}\vec{r}(t) = \int_a^t \frac{d^2}{dt^2}\vec{r}(t)dt + \frac{d}{dt}\vec{r}(a)$$

Find all curves that share our helix's acceleration vector.

# *Emergency Slide: Projectile Motion*

Do in class....

## *Arclength*

$$s(t) = \int_a^t |\frac{d}{dt}\vec{r}(t)|dt$$

is the distance traveled as $t$ went from $a$ to $t$ (watch out we are using the upper-limt = variable of integration!). Viewing a curve as parameterize by arclength means view is as a function of $s$.

View our helix as parameterized by arc length.

# *Curvature*

The curvature is $\kappa$ such that

$$\frac{d\hat{T}}{ds} = \kappa(s)\hat{N}$$

of

$$\left|\frac{d\hat{T}}{dt}\frac{dt}{ds}\right| = \left|\frac{d\hat{T}}{dt}\right|\left|\frac{dt}{ds}\right| = \frac{\left|\frac{d\hat{T}}{ds}\right|}{\left|\frac{d\vec{r}}{dt}\right|}$$

Find the curvature of our helix.

# *Frenet's Formulas*

$$\begin{aligned}\frac{d\hat{T}}{ds} &= \kappa\hat{N} \\ \frac{d\hat{N}}{ds} &= -\kappa\hat{T} + \tau\hat{B} \\ \frac{d\hat{B}}{ds} &= -\tau\hat{N}\end{aligned}$$

$\kappa$ is called the curve's *curvature* and $\tau$ is called the curve's *torsion*.

Summarize of knowledge of the helix. Generalize to a general helix.

# *An Example*

Suppose we have a monstrous wheel of radius 1 meter, which we imagine rolling a rate of 1 revolution per second without slipping along the x-axis in the $x$,$y$-plane.

1. Find a formula for the position of a piece of gum attached to the circumference of the wheel which at time zero is on the wheel's bottom (this is curve traced out is called a *cycloid*).
2. Find our gum's velocity at each time.
3. Describe the curves that share our gum's velocity vector at each time.
4. Find the distance traversed by our gum at each time $< 1/2$ (why?).