Homework due 3/11

1. $$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 4 \\ 1 & -2 & -3 \end{vmatrix} = (2\cdot(-3)-4\cdot(-2))\vec{i} - (3\cdot(-3)-4\cdot 1)\vec{j} + (3\cdot(-2)-2\cdot 1)\vec{k}$$
$$= 2\vec{i} + 13\vec{j} - 8\vec{k}$$

Then $\vec{a}\cdot(\vec{a}\times\vec{b}) = (3\vec{i}+2\vec{j}+4\vec{k})\cdot(2\vec{i}+13\vec{j}-8\vec{k}) = 3\cdot 2+2\cdot 13-4\cdot 8 = 0$

$\vec{b}\cdot(\vec{a}\times\vec{b}) = (\vec{i}-2\vec{j}-3\vec{k})\cdot(2\vec{i}+13\vec{j}-8\vec{k}) = 1\cdot 2-2\cdot 13+3\cdot 8 = 0$

so $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

2. $$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & t^2 & t^3 \\ 1 & 2t & 3t^2 \end{vmatrix} = (t^2\cdot 3t^2 - t^3\cdot 2t)\vec{i} - (t\cdot 3t^2 - t^3\cdot 1)\vec{j} + (t\cdot 2t - t^2\cdot 1)\vec{k}$$
$$= t^4\vec{i} - 2t^3\vec{j} + t^2\vec{k}.$$

Then $\vec{a}\cdot(\vec{a}\times\vec{b}) = (t\vec{i}+t^2\vec{j}+t^3\vec{k})\cdot(t^4\vec{i}-2t^3\vec{j}+t^2\vec{k}) = t\cdot t^4 - 2t^3\cdot t^2 + t^3\cdot t^2 = 0$

$\vec{b}\cdot(\vec{a}\times\vec{b}) = (\vec{i}+2t\vec{j}+3t^2\vec{k})\cdot(t^4\vec{i}-2t^3\vec{j}+t^2\vec{k}) = t^4 - 2t\cdot 2t^3 + 3t^2\cdot t^2 = 0.$

so $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

3. (a.) Meaningful. A vector "dotted with" a vector is a scalar.

(b.) Meaningless. $\vec{b}\cdot\vec{c}$ is a scalar, so cannot be "crossed with" $\vec{a}$.

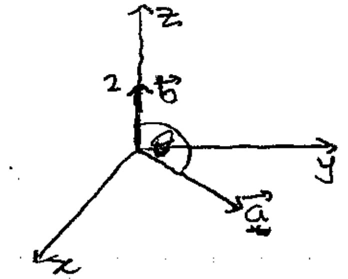
(c.) Meaningful. $\vec{b}\times\vec{c}$ is a vector, then so is $\vec{a}\times(\vec{b}\times\vec{c})$.

(d.) Meaningless. See (b.)

(e.) " "

(f.) Meaningful. This is the dot product of two vectors, which is thus a scalar.

4. (a) Let θ be the angle between $\vec{a}$ and $\vec{b}$, i.e. $\theta = 90°$.



Thus $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\cdot \sin 90° = 3\cdot 2\cdot 1 = 6.$

(b) $\vec{a}\times\vec{b}$ is orthogonal to $\vec{b}$, so the z-component of $\vec{a}\times\vec{b}$ is 0.
As drawn, $\vec{a}$ lies in the 1st quadrant of the xy-plane. Thus by the right-hand rule the x-component will be positive and the y-component negative (i.e. $\vec{a}\times\vec{b}$ is in the 4th quadrant).

5. $$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 0 & 0 & -4 \end{vmatrix} = -4\vec{i} - (-1)\cdot(-4)\vec{j} + 0\,\vec{k} = -4\vec{i} - 4\vec{j}$$

so $$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ -4 & -4 & 0 \end{vmatrix} = (0-(2\cdot(-4)))\vec{i} - 2\cdot(-4)\vec{j} + (-4-3\cdot(-4))\vec{k}$$
$$= 8\vec{i} + 8\vec{j} + 8\vec{k}$$

But $(\vec{a}\times\vec{b})\times\vec{c}$ is orthogonal to $\vec{c} = \langle 0, 0, -4\rangle$, so the $\vec{k}$-component of $(\vec{a}\times\vec{b})\times\vec{c}$ must equal 0. $\therefore (\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$.

[Alternatively, you may simply compute $(\vec{a}\times\vec{b})\times\vec{c}$.]

6. ~~The vectors~~

Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{k}$.

It is always true that $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$, and $\pm \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is a unit vector, so these are the two vectors we should use.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = (1\cdot1-0\cdot1)\vec{i} - (1\cdot1-1\cdot2)\vec{j} + (1\cdot0-1\cdot2)\vec{k}$$

$$= \vec{i} + \vec{j} - 2\vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{1^2+1^2+(-2)^2} = \sqrt{6}$$

so the vectors are $\frac{1}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} - \frac{2}{\sqrt{6}}\vec{k}$

and $-\frac{1}{\sqrt{6}}\vec{i} - \frac{1}{\sqrt{6}}\vec{j} + \frac{2}{\sqrt{6}}\vec{k}$.

7. $$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = (1\cdot5-2\cdot(-2))\vec{i} - (0\cdot5-2\cdot4)\vec{j} + (0\cdot5-2\cdot4)\vec{k}$$

$$= 9\vec{i} + 8\vec{j} - 8\vec{k} = \langle 9, 8, -8 \rangle$$

so $\vec{a}\cdot(\vec{b}\times\vec{c}) = \langle 6, 3, -1\rangle \cdot \langle 9, 8, -8 \rangle = 6\cdot9 + 3\cdot8 + (-1)\cdot(-8)$

$= 86$

so $V = |\vec{a}\cdot(\vec{b}\times\vec{c})| = |86| = 86$.

8. $\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} = \langle -1\cdot 2 - 0\cdot 3, -(1\cdot 2 - \overset{0}{\cancel{\ }}\cdot 7), 1\cdot 3 - (-1)\cdot 7\rangle$

$= \langle -2, -2, 10\rangle$

so $\vec{a}\cdot(\vec{b}\times\vec{c}) = \langle 2,3,1\rangle\cdot\langle -2,-2,10\rangle = 2\cdot(-2)+3\cdot(-2)+1\cdot 10 = 0$

so $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

9. Let $\vec{a} = \overrightarrow{PQ} = \langle 2-1, 4, 6-1\rangle = \langle 1,4,5\rangle$

$\vec{b} = \overrightarrow{PR} = \langle 3-1, -1, 2-1\rangle = \langle 2,-1,1\rangle$

$\vec{c} = \overrightarrow{PS} = \langle 6-1, 2, 8-1\rangle = \langle 5,2,7\rangle$

If P,Q,R,S are coplanar, then $\vec{a},\vec{b},\vec{c}$ are coplanar, and conversely, because any plane containing two of the points is parallel to the vector joining them, so if the points all lie in the same plane then so must all the vectors that join them; three distinct vectors are more than enough to define a plane, so the converse holds.

Now $\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 5 & 2 & 7 \end{vmatrix} = (-1\cdot 7 - 1\cdot 2)\vec{i} - (2\cdot 7 - 1\cdot 5)\vec{j} + (\overset{2\cdot 2-(-1)\cdot 5}{\cancel{\ }})\vec{k}$

$= -9\vec{i} - 9\vec{j} + 9\vec{k}$

$\vec{a}\cdot(\vec{b}\times\vec{c}) = \langle 1,4,5\rangle\cdot\langle -9,-9,9\rangle = 1\cdot(-9)+4\cdot(-9)+5\cdot 9 = 0$

so $\vec{a},\vec{b},\vec{c}$ are coplanar, and thus P,Q,R,S are also.

10. $\vec{r} = (0,0,0) + t\langle 2,-1,3\rangle = t\langle 2,-1,3\rangle$

The parametric equations are

$$x = 2t$$
$$y = -t$$
$$z = 3t$$

11. A direction vector is $\langle 6,1,-3\rangle - \langle 2,4,5\rangle = \langle 4,-3,-8\rangle$.

so a vector equation for the line is

$$\vec{r} = \langle 2,4,5\rangle + t\langle 4,-3,-8\rangle$$

thus parametric eqns are:

$$x = 2+4t$$
$$y = 4-3t$$
$$z = 5-8t$$

so symmetric eqns are:

$$\frac{x-2}{4} = \frac{y-4}{-3} = \frac{z-5}{-8}$$

12. A vector perpendicular to both $\vec{i}+\vec{j}$ and $\vec{j}+\vec{k}$

is $(\vec{i}+\vec{j})\times(\vec{j}+\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \vec{i} - \vec{j} + \vec{k}$

so the line is $\vec{r} = \langle 2,1,0\rangle + t\langle 1,-1,1\rangle$

i.e. $x = 2+t, \; y = 1-t, \; z = t$

The corresponding symmetric eqns are

$$x-2 = \frac{y-1}{-1} = z$$

13. The line $x+2=\frac{1}{2}y=z-3$ has direction vector

$$\langle a,b,c\rangle=\langle 1,2,1\rangle,$$

so the line with this direction vector that passes through $\langle 1,-1,1\rangle$ is

$$x=1+t,\quad y=-1+2t,\ z=1+t$$

with symmetric equations

$$x-1=\frac{y+1}{2}=z-1.$$

14. The direction vector of the 1st line is $\langle 4,1,-1\rangle-\langle 2,5,3\rangle=\langle 2,-4,-4\rangle$

" " " " 2nd line is $\langle 5,1,4\rangle-\langle -3,2,0\rangle=\langle 8,-1,4\rangle$

$$\langle 2,-4,-4\rangle\cdot\langle 8,-1,4\rangle=2\cdot 8+(-4)\cdot(-1)-4\cdot 4=\cancel{36}\ 4$$

$4\neq 0$ so the direction vectors are not perpendicular,

thus the lines are not perpendicular.