Homework #n-1 Due 11/28/04

#1 Find the points on the ellipsoid $x^2+2y^2+3z^2=1$ where the tangent plane is parallel to the plane $3x-y+3z=1$

$\nabla f(x_0,y_0,z_0)=\langle 2x_0, 4y_0, 6z_0\rangle$ and $\langle 3,-1,3\rangle$ are both normal to the ellipsoid at (x_0,y_0,z_0) where (x_0,y_0,z_0) is a point where the tangent plane is parallel to $3x-y+3z=1$

So we need $\langle 2x_0, 4y_0, 6z_0\rangle = c\langle 3,-1,3\rangle$

$\Leftrightarrow \quad \langle x_0, 2y_0, 3z_0\rangle = k\langle 3,-1,3\rangle$

So $x_0=3k$

$y_0=\frac{-1}{2}k$ & $x_0^2+2y_0^2+3z_0^2=1$

$z_0=k$

$\Rightarrow (3k)^2+2\left(\frac{-k}{2}\right)^2+3(k)^2 = k^2\left(9+\frac{1}{2}+3\right)=1$

$\Rightarrow k=\pm\frac{\sqrt{2}}{5}$

$\Rightarrow (x_0,y_0,z_0)=\left(\pm\frac{3\sqrt{2}}{5}, \mp\frac{1}{5\sqrt{2}}, \pm\frac{\sqrt{2}}{5}\right)$

#2 Suppose $(1,1)$ is a critical point of a function f with continuous second derivatives. In each case, what can you say about f?

(a) $f_{xx}(1,1)=4$ $f_{xy}(1,1)=1$ $f_{yy}(1,1)=2$

$D(1,1)=f_{xx}(1,1)f_{yy}(1,1)-[f_{xy}(1,1)]^2=4\cdot 2-1^2=7>0$

& $f_{xx}(1,1)>0$ so by the 2nd derivatives test f has a local minimum at $(1,1)$

(b) $f_{xx}(1,1)=4$ $f_{xy}(1,1)=3$ $f_{yy}(1,1)=2$

$D(1,1)=f_{xx}(1,1)f_{yy}(1,1)-[f_{xy}(1,1)]^2=4\cdot 2-3^2=-1<0$

$\Rightarrow f$ has a saddle point at $(1,1)$ by the 2nd derivatives test

#3 Use the level curves in the figure to predict the location of the critical points of f and whether f has a saddle point or a local extrema @ each of those points. Explain & check w/ 2^{nd} derivatives test.

As we move away from $(-1,1)$ & $(-1,-1)$ in any direction, the values of f are increasing, so we expect local minima.
As we move away from $(1,0)$ in any direction, the values of f are decreasing, so we expect a local maximum. There are hyperbola-shaped level curves near $(-1,0)$, $(1,1)$ & $(1,-1)$ and the values of f are decreasing as we move away in some directions and increases in others, so we expect saddle points.

$f(x,y) = 3x - x^3 - 2y^2 + y^4 \Rightarrow f_x(x,y) = 3 - 3x^2 \quad f_y = -4y + 4y^3$

$3 - 3x^2 = 0 \Rightarrow x = \pm 1 \quad -4y + 4y^3 = 0 \Rightarrow y(y^2 - 1) = 0 \Rightarrow y = 0$ or $y = \pm 1$

So the critical points are $(\pm 1, 0)$ & $(\pm 1, \pm 1)$.

$f_{xx} = -6x \quad f_{xy} = 0 \quad f_{yy} = 12y^2 - 4$

$\Rightarrow D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (-6x)(12y^2 - 4) - 0^2 = -72xy^2 + 24x$

point	D	f_{xx}	Conclusion
$(1,0)$	$24 > 0$	$-6 < 0$	local max @ $(1,0)$
$(-1,0)$	$-24 < 0$		saddle point @ $(-1,0)$
$(1,1)$	$-48 < 0$		saddle point @ $(1,1)$
$(1,-1)$	$-48 < 0$		saddle point @ $(1,-1)$
$(-1,1)$	$48 > 0$	$6 > 0$	local min @ $(-1,1)$
$(-1,-1)$	$48 > 0$	$6 > 0$	local min @ $(-1,-1)$

#4-6 Find the local max & min values & saddle points of the function.

#4 $f(x,y) = x^4 + y^4 - 4xy + 2$ $\quad f_x = 4x^3 - 4y \quad f_y = 4y^3 - 4x$

$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2$

$f_x = 0 \Rightarrow y = x^3 \quad \Rightarrow f_y = 4x^9 - 4x$

$f_y = 0 \Rightarrow x(x^8 - 1) = 0 \quad \Rightarrow x = 0 \text{ or } x = \pm 1$ So critical points are $(0,0)\ (1,1)\ (-1,-1)$

$D(0,0) = 0 \cdot 0 - (-4)^2 = -16 < 0 \Rightarrow (0,0)$ is a saddle point

$D(1,1) = 12 \cdot 12 - (-4)^2 = 144 - 16 > 0 \quad f_{xx}(1,1) = 12 > 0$

$\Rightarrow (1,1)$ is a local min

$D(-1,-1) = 12 \cdot 12 - (-4)^2 = 144 - 16 > 0 \quad f_{xx}(-1,-1) = 12 > 0$

$\Rightarrow (-1,-1)$ is a local min

#5 $f(x,y) = xy(1-x-y) = xy - x^2y - xy^2$

$f_x = y - 2xy - y^2 \qquad f_y = x - x^2 - 2xy$

$f_{xx} = -2y \quad f_{xy} = 1 - 2x - 2y \quad f_{yy} = -2x$

$f_x = 0 \Rightarrow y = 0$ or $y = 1 - 2x \quad \Rightarrow f_y = x - x^2$ or $3x^2 - x = f_y$

$f_y = 0 \Rightarrow \begin{cases} x - x^2 = 0 \Rightarrow x = 0 \text{ or } x = 1 \\ \text{or } 3x^2 - x = 0 \Rightarrow x = 0 \text{ or } x = \frac{1}{3} \end{cases}$ So critical points are $(0,0)\ (1,0)$
$(0,1)\ (\frac{1}{3}\ \frac{1}{3})$

$D(0,0) = D(1,0) = D(0,1) = -1 < 0$

$\Rightarrow (0,0), (1,0), (0,1)$ are saddle points

$D(\frac{1}{3},\frac{1}{3}) = \frac{1}{3} \qquad f_{xx}(\frac{1}{3},\frac{1}{3}) = -\frac{2}{3} < 0$

$\Rightarrow f(\frac{1}{3},\frac{1}{3}) = \frac{1}{27}$ is a local maximum

#6 $f(x,y) = x^2 y e^{-x^2-y^2}$

$f_x = x^2 y e^{-x^2-y^2}(-2x) + 2xy\, e^{-x^2-y^2} = 2xy(1-x^2)e^{-x^2-y^2}$

$f_y = x^2 y e^{-x^2-y^2}(-2y) + x^2 e^{-x^2-y^2} = x^2(1-2y^2)e^{-x^2-y^2}$

$f_{xx} = 2y(2x^4-5x^2+1)e^{-x^2-y^2}$

$f_{xy} = 2x(1-x^2)(1-2y^2)e^{-x^2-y^2}$

$f_{yy} = 2x^2 y(2y^2-3)e^{-x^2-y^2}$

$f_x = 0 \Rightarrow x=0,\ y=0$ or $x=\pm 1$

$x=0 \Rightarrow f_y = 0\ \forall y$ So all $(0,y)$ are critical points.

$y=0 \Rightarrow f_y = 0 \Rightarrow x^2 e^{-x^2} = 0 \Rightarrow x=0$ so $(0,0)$ is a critical point

$x=\pm 1 \Rightarrow f_y = 0 \Rightarrow (1-2y^2)e^{-1-y^2} = 0 \Rightarrow y = \pm\frac{1}{\sqrt{2}}$

so $(1, \pm\frac{1}{\sqrt{2}})$ & $(-1, \pm\frac{1}{\sqrt{2}})$ are critical points

$D(0,y)=0$ so the second derivatives test tells us nothing, but when $y>0$, $x^2 y e^{-x^2-y^2} \geq 0$ and $=0$ only when $x=0$ So $f(0,y)=0$ is a local min when $y>0$ (Since everything around $f(0,y)=0$ is >0).

$y<0$, $x^2 y e^{-x^2-y^2} \leq 0$ and $=0$ only when $x=0$. So $f(0,y)=0$ is a local max when $y<0$ (Since everything around $f(0,y)=0$ is <0)

And $(0,0)$ is a saddle point

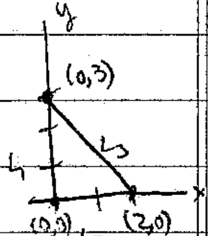
$D(\pm 1, \frac{1}{\sqrt{2}}) = 8e^{-3} > 0$ $f_{xx}(\pm 1, \frac{1}{\sqrt{2}}) = -2\sqrt{2}\, e^{-3/2} < 0$

so $f(\pm 1, \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} e^{-3/2}$ are local maxima

$D(\pm 1, -\frac{1}{\sqrt{2}}) = 8e^{-3} > 0$ $f_{xx}(\pm 1, -\frac{1}{\sqrt{2}}) = 2\sqrt{2}\, e^{-3/2} > 0$

so $f(\pm 1, -\frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}} e^{-3/2}$ are local minima

#7 Find the absolute maximum & minimum values of f on the set D.
$f(x,y)=1+4x-5y$ D is the closed triangular region with vertices $(0,0)$, $(2,0)$, and $(0,3)$



f is a polynomial so it is continuous on D and an absolute max and min exist. $f_x=4$ & $f_y=-5$ So there are no critical points inside D & the absolute extrema must live on the boundary.
$L_1=\overline{(0,0)\ (0,3)}$ $L_2=\overline{(0,0)\ (2,0)}$ $L_3=\overline{(2,0)\ (0,3)}$
On L_1, $x=0 \Rightarrow f(0,y)=1-5y$ $(0\le y\le 3)$ which is a decreasing function so the maximum is $f(0,0)=1$ & the minimum is $f(0,3)=-14$
On L_2, $y=0 \Rightarrow f(x,0)=1+4x$ $(0\le x\le 2)$ which is an increasing function so the maximum is $f(2,0)=9$ and the minimum is $f(0,0)=1$
On L_3, $y=-\frac{3}{2}x+3 \Rightarrow f(x,-\frac{3}{2}x+3)=\frac{23}{2}x-14$ $(0\le x\le 2)$
which is an increasing function so the maximum is $f(2,0)=9$ and the minimum is $f(0,3)=-14$
So the absolute maximum is $f(2,0)=9$ and the absolute minimum is $f(0,3)=-14$.

#8 Find three positive numbers whose sum is 100 & whose product is a maximum
$x+y+z=100$ So we want to maximize $f(x,y)=xy(100-x-y)=100xy-x^2y-xy^2$
$f_x=100y-2xy-y^2$ $f_y=100x-x^2-2xy$
$f_{xx}=-2y$ $f_{xy}=100-2x-2y$ $f_{yy}=-2x$
$f_x=0 \Rightarrow \begin{cases} y=0 & \text{\& } f_y=0 \Rightarrow x=0 \text{ or } x=100 \\ \text{or } y=100-2x & \text{\& } f_y=0 \Rightarrow 3x^2-100x=0 \Rightarrow x=0 \text{ or } x=\frac{100}{3} \end{cases}$
So the critical points are $(0,0)$, $(100,0)$, $(0,100)$, & $(\frac{100}{3},\frac{100}{3})$

$D(0,0) = D(100,0) = D(0,100) = -10{,}000 < 0$ so

$(0,0)$, $(100,0)$ & $(0,100)$ are saddle points

$D\left(\frac{100}{3}, \frac{100}{3}\right) = \frac{10{,}000}{3}$ $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$

So $\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum So $\boxed{x = y = z = \frac{100}{3}}$

#9 Find the dimensions of the rectangular box with largest volume if the total surface area is given as $64\ cm^2$.

Surface area $= 2(xy + yz + xz) = 64 cm^2$ So $xy + yz + xz = 32$

$\Rightarrow z = \frac{32 - xy}{x+y}$ So we want to maximize $f(x,y) = \left(\frac{32-xy}{x+y}\right)xy$

$f_x = \frac{32y^2 - 2xy^3 - x^2y^2}{(x+y)^2} = y^2\left(\frac{32 - 2xy - x^2}{(x+y)^2}\right)$

$f_y = x^2\left(\frac{32 - 2xy - y^2}{(x+y)^2}\right)$

$f_x = 0 \Rightarrow y = \frac{32 - x^2}{2x}$ (since y cannot $= 0$) Substituting

into $f_y = 0 \Rightarrow 32(4x^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0$

$\Rightarrow 3x^4 + 64x^2 - (32)^2 = 0$

$\Rightarrow x^2 = \frac{64}{6} \Rightarrow x = \frac{8}{\sqrt{6}} \Rightarrow y = \frac{64/3}{16/\sqrt{6}} = \frac{8}{\sqrt{6}}$

$\Rightarrow z = \frac{8}{\sqrt{6}}$. Thus the box is the cube with edge length $\frac{8}{\sqrt{6}}$ cm

#10 A cardboard box without a lid is to have a volume of $32{,}000\,cm^3$.
Find the dimensions that minimize the amount of cardboard used.
The surface area of the box is $xy + 2(xz + yz)$ & $xyz = 32{,}000$
$\Rightarrow z = \dfrac{32{,}000}{xy}$ So we wish to minimize
$f(x,y) = xy + \dfrac{64{,}000(x+y)}{xy} = xy + 64{,}000(x^{-1} + y^{-1})$
$f_x = y - 64{,}000x^{-2}$
$f_y = x - 64{,}000\,y^{-2}$
$f_x = 0 \Rightarrow y = \dfrac{64{,}000}{x^2}$ sub. into $f_y = 0 \Rightarrow x^3 = 64{,}000 \Rightarrow x = 40 \Rightarrow y = 40$

$D(x,y) = [(2)(64{,}000)]^2 x^{-3} y^{-3} - 1 > 0$ @ $(40, 40)$
$f_{xx}(40, 40) > 0$ so $f(40,40)$ is a minimum and the box dimensions are $x = y = 40cm$ $z = 20cm$