Homework due Mon. 11/22

① The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in & the height is 140 in?

$V_{cone} = \frac{\pi r^2 h}{3}$ so $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dr}{dh}$

$\frac{\partial V}{\partial r} = \frac{2\pi r h}{3}$ & $\frac{\partial V}{\partial h} = \frac{\pi r^2}{3}$

$\Rightarrow \frac{dV}{dt} = \frac{2\pi(120)(140)\,in^2}{3} \cdot \frac{1.8\,in}{s} + \frac{\pi(120)^2\,in^2}{3} \cdot \frac{-2.5\,in}{s}$

$= 20160\pi - 12000\pi = \boxed{8160\pi\ in^3/s}$

② Find the directional derivative $f(x,y,z) = x^2 + y^2 + z^2$ @ $P(2,1,3)$ in the direction of the origin.

$\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle \Rightarrow \nabla f(2,1,3) = \langle 4,2,6 \rangle$

$\overrightarrow{PO} = \langle -2,-1,-3 \rangle$ & $|\overrightarrow{PO}| = \sqrt{(-2)^2+(-1)^2+(-3)^2} = \sqrt{14}$

$\Rightarrow \vec{u} = \frac{1}{\sqrt{14}}\langle -2,-1,-3 \rangle \Rightarrow D_{\vec{u}} f(2,1,3) = \langle 4,2,6 \rangle \cdot \frac{1}{\sqrt{14}}\langle -2,-1,-3 \rangle$

$= \frac{1}{\sqrt{14}}(-8 + -2 + -18) = \frac{-28}{\sqrt{14}} = -2\sqrt{14}$

③-⑤ Find the maximum rate of change of f at the given point and the direction in which it occurs.

③ $f(x,y) = y^2/x$, $(2,4)$ $\nabla f(x,y) = \langle -y^2x^{-2}, 2yx^{-1} \rangle = \langle -y^2/x^2, 2y/x \rangle$

$\nabla f(2,4) = \langle -4, 4 \rangle \sim \langle -1, 1 \rangle$ $|\nabla f(2,4)| = \sqrt{(-4)^2+(4)^2} = \sqrt{32} = 4\sqrt{2}$

So the direction of the maximum rate of change is $\langle -1, 1 \rangle$ & the maximum rate of change is $4\sqrt{2}$.

④ $f(x,y,z) = x^2y^3z^4 \qquad (1,1,1)$

$\nabla f(x,y,z) = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle \qquad \nabla f(1,1,1) = \langle 2,3,4 \rangle$

$|\nabla f(1,1,1)| = \sqrt{2^2+3^2+4^2} = \sqrt{29}$

So the maximum rate of change is $\sqrt{29}$ in the direction of $\langle 2,3,4 \rangle$

⑤ $f(x,y,z) = \tan(x+2y+3z) \qquad (-5,1,1)$

$\nabla f(x,y,z) = \langle \sec^2(x+2y+3z)(1), \sec^2(x+2y+3z)(2), \sec^2(x+2y+3z)(3) \rangle$

$\nabla f(-5,1,1) = \langle \sec^2(0), 2\sec^2(0), 3\sec^2(0) \rangle = \langle 1,2,3 \rangle$

$|\nabla f(-5,1,1)| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

So the maximum rate of change is $\sqrt{14}$ in the direction of $\langle 1,2,3 \rangle$

⑥ ⓐ Show that a differentiable function f decreases most rapidly at $\vec{x}$ in the direction opposite to the gradient vector, i.e. in the direction of $-\nabla f(\vec{x})$.

From the proof of Thm. 15, $D_{\vec{u}}f = |\nabla f| \cos\theta$ When $\theta = \pi$, $\cos\theta = -1$, its minimum value. So the minimum value of $D_{\vec{u}}f$ is $-|\nabla f|$ occuring when $\theta = \pi$, or when $\vec{u}$ is in the opposite direction of ∇f.

ⓑ Find the direction in which $f(x,y) = x^4y - x^2y^3$ decreases fastest @ $P(2,-3)$.

$\nabla f(x,y) = \langle 4x^3y - 2xy^3, x^4 - 3x^2y^2 \rangle \qquad \nabla f(2,-3) = \langle 12, -92 \rangle$

So the direction in which $f(x,y)$ decreases the fastest @ $(2,-3)$ is $-\nabla f(2,-3) = \langle -12, 92 \rangle$

(7) Find the direction in which the directional derivative of $f(x,y)=x^2+\sin xy$ @ the point $(1,0)$ has the value 1.

$f_x = 2x + y\cos xy \quad f_y = x\cos xy$

$f_x(1,0) = 2+0\cos 0 = 2 \quad f_y(1,0) = 1\cos 0 = 1$

$D_{\vec{u}} f(1,0) = f_x(1,0)\cos\theta + f_y(1,0)\sin\theta = 2\cos\theta + \sin\theta$, where θ is the angle $\vec{u}$ makes w/ the positive x-axis.

We want $D_{\vec{u}} f(1,0) = 2\cos\theta + \sin\theta = 1 \Rightarrow \sin\theta = 1-2\cos\theta$

$\Rightarrow \sin^2\theta = 1-4\cos\theta+4\cos^2\theta \Rightarrow 1-\cos^2\theta = 1-4\cos\theta+4\cos^2\theta$

$\Rightarrow 5\cos^2\theta - 4\cos\theta = 0 \Rightarrow \cos\theta(5\cos\theta-4)=0$

$\Rightarrow \cos\theta = 0$ or $5\cos\theta - 4 = 0 \Rightarrow \cos\theta = 4/5$

$\Rightarrow \theta = \pi/2$ or $\theta = \cos^{-1}(4/5) \approx 5.64$

(8) (a) Find the rate of change of T @ $(1,2,2)$ in the direction toward the point $(2,1,3)$.

$T = \dfrac{K}{\sqrt{x^2+y^2+z^2}}$ & $120 = T(1,2,2) = \dfrac{K}{3} \Rightarrow K = 360.$

$\vec{u} = \dfrac{\langle 2-1,\ 1-2,\ 3-2\rangle}{\sqrt{(2-1)^2+(1-2)^2+(3-2)^2}} = \dfrac{\langle 1,-1,1\rangle}{\sqrt{3}}$

$\nabla T = -360\cdot\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\langle \cancel{2}x, \cancel{2}y, \cancel{2}z\rangle$

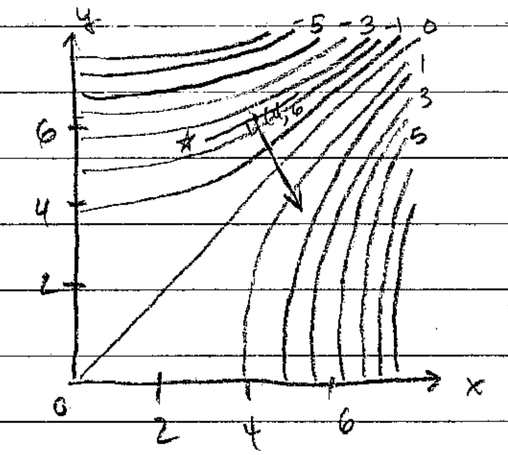
$\nabla T\cdot\vec{u} = \dfrac{-40}{3}\langle 1,2,2\rangle\cdot\dfrac{1}{\sqrt{3}}\langle 1,-1,1\rangle = \dfrac{-40}{3\sqrt{3}}(1-2+2) = \dfrac{-40}{3\sqrt{3}}$

(b) Show that at any point in the ball the direction of increase in temperature is given by a vector that points toward the origin.

$\langle x,y,z\rangle$ is the position vector of any point on the ball, so $\langle -x,-y,-z\rangle$ always points toward the origin.

So $\nabla T = \dfrac{360}{(x^2+y^2+z^2)^{3/2}}\langle -x,-y,-z\rangle$ also points toward the origin.

(9) Sketch the gradient vector $\nabla f(4,6)$ for the function f whose level curves are shown. Explain how you chose the direction and length of this vector.



We know that $\nabla f(4,6)$ is perpendicular to the level curve that includes $(4,6)$. We estimate a portion of this level curve using the others. Then $\nabla f(4,6)$ is perpendicular to this line (*). We need to estimate its length. If we look at where $\nabla f(4,6)$ would intersect the level curves -2 & -3 if we extended it, we would see that the two points of intersection are about .5 apart. So the rate of change is approximately $\frac{-2-(-3)}{1/2} = \frac{1}{1/2} = 2$. So our vector has length ≈ 2.

(10) & (11) Find the equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point

(10) $x^2+2y^2+3z^2=21$, $(4,-1,1)$

(a) Let $F(x,y,z) = x^2+2y^2+3z^2$ (So $x^2+2y^2+3z^2=21$ is just a level set @ $F=21$.)

$F_x = 2x$ $\quad F_y = 4y$ $\quad F_z = 6z$ $\quad F_x(4,-1,1)=8$ $\quad F_y(4,-1,1)=-4$ $\quad F_z(4,-1,1)=6$

So our tangent plane @ $(4,-1,1)$ is

$8(x-4) + -4(y-1) + 6(z-1) = 0 \Rightarrow 8x-32-4y-4+6z-6=0$

$\Rightarrow 8x-4y+6z=42 \Rightarrow 4x-2y+3z=21$

(b) From equation 20, we have the symmetric equations of the normal line:

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)} \Rightarrow \frac{x-4}{8} = \frac{y+1}{-4} = \frac{z-1}{6}$$

$$\Rightarrow \frac{x-4}{4} = \frac{y+1}{-2} = \frac{z-1}{3}$$

(11) $x - z = 4\arctan(yz)$ $\quad (1+\pi, 1, 1)$

(a) Let $F(x,y,z) = x - z - 4\arctan(yz)$ (Then $x-z=4\arctan(yz)$ is a level surface at $F=0$.

$$\nabla F(x,y,z) = \left\langle 1, \frac{-4}{1+y^2z^2}, -1 - \frac{4}{1+y^2z^2} \right\rangle$$

$$\nabla F(1+\pi,1,1) = \left\langle 1, \frac{-4}{1+1}, -1 - \frac{4}{1+1} \right\rangle = \langle 1, -2, -3 \rangle$$

So the equation for the tangent plane is

$$1(x-(1+\pi)) + -2(y-1) + -3(z-1) = 0$$

$$\Rightarrow x - 1 - \pi - 2y + 2 - 3z + 3 = 0 \Rightarrow x - 2y - 3z = \pi - 4$$

(b) So the normal line has symmetric equations:

$$x - 1 - \pi = \frac{y-1}{-2} = \frac{z-1}{-3}$$