Math 8
HW Due 10/29/09

1) Find the Taylor polynomial $T_n(x)$ for the function f at the number a.

$f(x) = \sin x \qquad a = \frac{\pi}{6} \qquad n = 3$

$f(x) = \sin x \Rightarrow f(\frac{\pi}{6}) = \frac{1}{2}$

$f'(x) = \cos x \Rightarrow f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x \Rightarrow f''(\frac{\pi}{6}) = -\frac{1}{2}$

$f'''(x) = -\cos x \Rightarrow f'''(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$

Thus

$$T_3(x) = f(\tfrac{\pi}{6}) + \frac{f'(\frac{\pi}{6})}{1!}(x - \tfrac{\pi}{6}) + \frac{f''(\frac{\pi}{6})}{2!}(x - \tfrac{\pi}{6})^2 + \frac{f'''(\frac{\pi}{6})}{3!}(x - \tfrac{\pi}{6})^3$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \tfrac{\pi}{6}) - \frac{1}{4}(x - \tfrac{\pi}{6})^2 - \frac{\sqrt{3}}{12}(x - \tfrac{\pi}{6})^3$$

2) Use the information in problem 1 to estimate $\sin 35°$ correct to five decimal places

First observe $35° = 30° + 5° = \frac{\pi}{6} + \frac{\pi}{36}$ $\left(\frac{30}{6} = 5 \Rightarrow 5° = \frac{\pi/6}{6} = \frac{\pi}{36}\right)$

Next we will check that the error of T_3 at $35°$ is less than 10^{-5}. Now R_3 = error of T_3

and $R_3 \leq \frac{M}{4!}\left(\frac{\pi}{6} + \frac{\pi}{36} - \frac{\pi}{6}\right)^4$

so it remains to find M. Now

$f(x) = \sin x$ giving $f^{(4)}(x) = \sin x$

Thus $|f^{(4)}(x)| = |\sin x| \leq 1$

Hence we can take $M = 1$

and $R_3 \leq \frac{1}{4!}\left(\frac{\pi}{36}\right)^4 \approx 2.4165 \times 10^{-6}$

So our estimate with T_3 will be correct to 5 decimal places.

$$T_3\left(\frac{\pi}{6} + \frac{\pi}{36}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right) - \frac{1}{4}\left(\frac{\pi}{36}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3 \approx 0.57358$$

3) Which of the following expressions are meaningful? Which are meaningless? Explain

(a) $(a \cdot b) \cdot c$ (b) $(a \cdot b) c$ (c) $|a|(b \cdot c)$

(d) $a \cdot (b + c)$ (e) $a \cdot b + c$ (f) $|a| \cdot (b + c)$

(a) is meaningless because $(a \cdot b)$ is a scalar not a vector and the dot product between a vector and a scalar is not defined

(b) meaningful $(a \cdot b)$ is a scalar and multiplying a vector and a scalar is defined

(c) meaningful $(b \cdot c)$ is a scalar and $|a|$ is a scalar and multiplication between scalars is defined.

(d) meaningful a is a vector and $b + c$ is a vector and the dot product of two vectors is defined

(e) meaningless $a \cdot b$ is a scalar, c is a vector the addition of a scalar and a vector is not defined

(f) meaningless $|a|$ is a scalar, $b + c$ is a vector the dot product of a scalar and a vector is not defined.

4) Find $a \cdot b$ $\quad a = 4j - 3k$, $\quad b = 2i + 4j + 6k$

$= 0i + 4j - 3k$

$a \cdot b = (a_1 i + a_2 j + a_3 k) \cdot (b_1 i + b_2 j + b_3 k)$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$

$= (0)(2) + (4)(4) + 6(-3) = 16 - 18 = -2$

5) Find $a \cdot b$ if $|a| = 12$ $|b| = 15$ and the angle between a and b is $\frac{\pi}{6}$

Using Theorem $\boxed{3}$ we know $a \cdot b = |a||b|\cos\theta$

thus $a \cdot b = (12)(15) \cdot \cos\frac{\pi}{6} = (12)(15)\frac{\sqrt{3}}{2}$

$= (6)(15)(\sqrt{3}) = 90\sqrt{3}$

6) Find the angle between the vectors

$a = 2i - j + k \qquad b = 3i + 2j - k$

By corollary $\boxed{6}$ $\cos\theta = \frac{a \cdot b}{|a||b|}$

$a \cdot b = 2(3) + (-1)(2) + (-1)(1) = 6 - 2 - 1 = 3$

$|a| = \sqrt{2^2 + (-1)^2 + (1)^2} = \sqrt{6} \qquad |b| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$

thus

$\cos\theta = \frac{3}{\sqrt{6}\sqrt{14}} = \frac{3}{\sqrt{6 \cdot 14}} = \frac{3}{\sqrt{2 \cdot 3 \cdot 2 \cdot 7}} = \frac{3}{2\sqrt{21}} = \frac{3\sqrt{21}}{42} = \frac{\sqrt{21}}{14}$

Thus $\theta = \cos^{-1}\left(\frac{\sqrt{21}}{14}\right) \approx 1.23732$

7) Determine whether the given angles are orthogonal, parallel or neither

(a) $a = \langle -5, 3, 7 \rangle$ $b = \langle 6, -8, 2 \rangle$

(b) $a = \langle 4, 6 \rangle$ $b = \langle -3, 2 \rangle$

(c) $a = -i + 2j + 5k$ $b = 3i + 4j - k$

(d) $a = 2i + 6j - 4k$ $b = -3i - 9j + 6k$

(a) $a \cdot b = (-5)(6) + 3(-8) + 2(7) = -30 - 24 + 14 = -40 \neq 0$

thus the vectors are not orthogonal

$\cos\theta = \frac{a \cdot b}{|a||b|}$ $|a| = \sqrt{(-5)^2 + (3)^2 + 7^2} = \sqrt{25 + 9 + 49} = \sqrt{83}$

$|b| = \sqrt{6^2 + (-8)^2 + 2^2} = \sqrt{36 + 64 + 4} = \sqrt{104}$

$\cos\theta = \frac{-40}{\sqrt{104}\sqrt{83}} \neq 1$ or -1 thus a, b are not parallel

also we can see $\nexists$ $r \in \mathbb{R}$ s.t $ra = b$

thus a' and b' are not parallel

(b) $a \cdot b = 4(-3) + 6(2) = -12 + 12 = 0$

Thus a and b are orthogonal

(c) $a \cdot b = (-1)(3) + 2(4) + (-1)(5) = -3 + 8 - 5 = 0$

Thus a and b are orthoginal

(d) $a \cdot b = (2)(-3) + 6(-9) + 6(-4) = -6 - 54 - 24 = -84 \neq 0$

Thus a and b are not orthoginal

but $-\frac{3}{2}a = (\frac{-3}{2})2i + (\frac{-3}{2})6j - (\frac{-3}{2})4k = -3i - 9j + 6k = b$

Thus a and b are parallel.

8) Find two unit vectors that make an angle of $60°$ with $v = \langle 3, 4 \rangle$

if a is a unit vector that makes an angle of $60°$ with v

then $a \cdot v = |a|\,|v| \cos 60° = \frac{|v|}{2}$.

but $|v| = \sqrt{3^2 + 4^2} = 5$

hence if $a = \langle a_1, a_2 \rangle$

then $a \cdot v = 3a_1 + 4a_2 = \frac{5}{2}$ $(*)$

but we also know $a_1^2 + a_2^2 = 1$

from $(*)$ we have $a_2 = \frac{5}{8} - \frac{3}{4} a_1$

putting this into $a_1^2 + a_2^2 = 1$

we get $a_1^2 + a_2^2 = a_1^2 + \left(\frac{5}{8} - \frac{3}{4} a_1\right)^2 = 1$

$$a_1^2 + \frac{25}{64} - \frac{15}{16} a_1 + \frac{9}{16} a_1^2 = 1$$

$$\frac{25}{16} a_1^2 - \frac{15}{16} a_1 + \frac{25}{64} - 1 = 0$$

$$64\left(\frac{25}{16} a_1^2 - \frac{15}{16} a_1 + \frac{25}{64} - 1\right) = 64(0)$$

$$100 a_1^2 - 60 a_1 + 25 - 64 = 0$$

$$100 a_1^2 - 60 a_1 - 39 = 0$$

by the quadratic formula we get

$$a_1 = \frac{60 \pm \sqrt{60^2 - 4(100)(-39)}}{200} = \frac{3}{10} \pm \frac{\sqrt{3600 + 4(100)(39)}}{200}$$

$$= \frac{3}{10} \pm \frac{10\sqrt{36 + 4(39)}}{200} = \frac{3}{10} \pm \frac{2\sqrt{9 + 39}}{20}$$

$$= \frac{3}{10} \pm \frac{\sqrt{3(3 + 13)}}{10} = \frac{3}{10} \pm \frac{\sqrt{16 \cdot 3}}{10} = \frac{3}{10} \pm \frac{4\sqrt{3}}{10}$$

if $a_1 = \frac{3+4\sqrt{3}}{10}$ then $a_2 = \frac{5}{8} - \frac{3}{4}\left(\frac{3+4\sqrt{3}}{10}\right)$

$= \frac{25-9-12\sqrt{3}}{40} = \frac{16-12\sqrt{3}}{40} = \frac{4-3\sqrt{3}}{10}$

giving $\left\langle \frac{3+4\sqrt{3}}{10}, \frac{4-3\sqrt{3}}{10} \right\rangle$

similarly if

$a_1 = \frac{3-4\sqrt{3}}{10}$ then $a_2 = \frac{5}{8} - \frac{3}{4}\left(\frac{3-4\sqrt{3}}{10}\right)$

$= \frac{25-9+12\sqrt{3}}{40} = \frac{4+3\sqrt{3}}{10}$

giving $\left\langle \frac{3-4\sqrt{3}}{10}, \frac{4+3\sqrt{3}}{10} \right\rangle$ $\boxed{4}$

9) Find the scalar and vector projections of b onto a

$a = \langle 3, -4 \rangle, \quad b = \langle 5, 0 \rangle$

Scalar projection

$$\text{comp}_a b = \frac{a \cdot b}{|a|}$$

vector projection

$$\text{proj}_a b = \frac{a \cdot b}{|a|^2} a$$

$a \cdot b = 3(5) + (-4)(0) = 15$

$|a| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$

$|a|^2 = 25$

Thus $\text{comp}_a b = \dfrac{15}{5} = 3$

$$\text{proj}_a b = \frac{15}{25} \langle 3, -4 \rangle = \frac{3}{5} \langle 3, -4 \rangle = \left\langle \frac{9}{5}, \frac{-12}{5} \right\rangle$$

10) Find the scalar and vector projections of b onto a

$a = \langle 4, 2, 0 \rangle \quad b = \langle 1, 1, 1 \rangle$

$a \cdot b = 4(1) + 2(1) + 0(1) = 6$

$|a| = \sqrt{(4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$

$|a|^2 = 20$

Thus $\text{comp}_a b = \frac{a \cdot b}{|a|} = \frac{6}{2\sqrt{5}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$

$\text{proj}_a b = \frac{a \cdot b}{|a|^2} a = \frac{6}{20} \langle 4, 2, 0 \rangle = \langle \frac{6}{5}, \frac{6}{10}, 0 \rangle$

11) If $a = \langle 3, 0, -1 \rangle$, find a vector b such that $\text{comp}_a b = 2$

Now $\text{comp}_a b = \frac{a \cdot b}{|a|}$

Now $|a| = \sqrt{(3)^2 + (0)^2 + (-1)^2} = \sqrt{10}$

so if $\text{comp}_a b = 2$ then $a \cdot b = 2\sqrt{10}$

if $b = \langle b_1, b_2, b_3 \rangle$

$a \cdot b = 3b_1 - b_3$

So if we take $b = \langle 0, 0, -2\sqrt{10} \rangle$

then $a \cdot b = 3(0) - (-2\sqrt{10}) = 2\sqrt{10}$

which implies

$\text{comp}_a b = \frac{2\sqrt{10}}{\sqrt{10}} = 2$ as desired.

Note: In general if $3b_1 - b_3 = 2\sqrt{10}$

take $t = b_3$ then $b_1 = \frac{2}{3}\sqrt{10} - t/3$

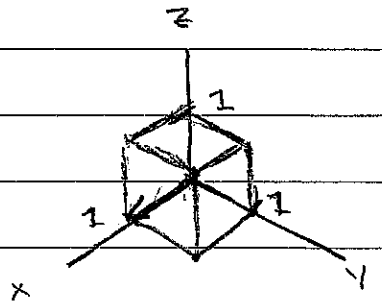
Then any element of the form $\langle \frac{2}{3}\sqrt{10} - \frac{t}{3}, s, t \rangle$ where $s, t \in \mathbb{R}$

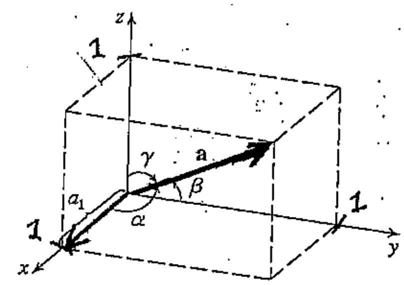
gives the desired result

ie $a \cdot b = 3\left(\frac{2}{3}\sqrt{10} - t/3\right) + t = 2\sqrt{10}$

so $\text{comp}_a b = \frac{2\sqrt{10}}{\sqrt{10}} = 2$

12) Find the angle between the diagonal of a cube and one of its edges.





Drawing the cube in the first quadrant of $\mathbb{R}^3$ we can see the diagonal of the cube is given by the vector $\langle 1,1,1 \rangle$

one of its edges is given by the vector $\langle 1,0,0 \rangle$

by corollary [6]

$$\cos\theta = \frac{a \cdot b}{|a||b|} \qquad \text{let } a = \langle 1,1,1 \rangle, \quad b = \langle 1,0,0 \rangle$$

then $a \cdot b = 1(1) + 1(0) + 1(0) = 1$

$$|a| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$|b| = \sqrt{1^2+0^2+0^2} = 1$$

Thus $\cos\theta = \dfrac{1}{1 \cdot \sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$$