LECTURE OUTLINE
Cross Product

Professor Leibon

Math 8

Nov. 1, 2004

Goals

Cross Product

Parallelogram Area

Parallelepiped Volume

Review

Let $\vec{a} =< -1, 2, 5 >$ and $\vec{b} =< 2, 2, 7 >$. Find a length 3 vector such that its component in the $\vec{b}$ is 2. What is your vectors component in the $\vec{a}$ direction? Is it possible to find a length 3 vector such that its component in the $\vec{b}$ is 2 which is perpendicular to $\vec{a}$?

Cross Product

Given vectors $\vec{a}$ and $\vec{b}$ we define $\vec{a} \times \vec{b}$ to be the unique vector satisfying

(1) $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and to $\vec{b}$ (or zero).

(2) It has length equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

(3) $\vec{a} \times \vec{b}$ is in the direction determined by the right hand rule going from $\vec{a}$ to $\vec{b}$.

Example: Let $\vec{a} =< -7, 0, 0 >$ and $\vec{b} =< 0, 0, 2 >$ and find $\vec{a} \times \vec{b}$.

Main Theorem

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \times \vec{b}$ equals

$$(a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}.$$

Example: Let $\vec{a} = < -1, 2, 5 >$ and $\vec{b} = < 2, 2, 7 >$, find $\vec{a} \times \vec{b}$. Find the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Prove property (1).

Basic Properties

1. $\vec{a} \times \vec{b} = -\vec{a} \times \vec{b}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
4. $\vec{c} \times (\vec{a} + \vec{b}) = \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$
5. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Example: Let $\vec{a} =< -1, 2, 5 >$ and $\vec{b} =< 2, 2, 7 >$, $\vec{c} =< 1, 0, 0 >$ and find $(\vec{a} + 3\vec{c} + 7\vec{b}) \times (\vec{c} + 3\vec{b})$.

Warm Up: Parallelogram Area in the plane

The oriented area A of the parallelogram determined by $\vec{a} = a_1\hat{i} + a_2\hat{j}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j}$ satisfies

$$A^2 = (base)^2(height)^2 = |\vec{a}|^2|\vec{b}|^2(\sin(\theta))^2$$

$$= (a_1^2+a_2^2)(b_1^2+b_2^2)(1-\cos(\theta)^2) = ((a_1^2+a_2^2)(b_1^2+b_2^2)-(a_1b_1+a_2b_2)^2)$$

$$= (a_1b_2 - a_2b_1)^2 = |\vec{a} \times \vec{b}|^2$$

Hence the square root, *the oriented area*, is given by

$$a_1b_2 - a_2b_1 \equiv \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \equiv \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

(The last line is "absolutely" the **stupidest notation ever introduced.** Why?)

Using this notation

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Let $\vec{b} =< 2, 2, 7 >$ and $\vec{c} =< 3, 2, -1 >$. Find $\vec{b} \times \vec{c}$.

Parallelepiped Volume

The oriented volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$ is given by

$$\vec{a} \cdot (\vec{b} \times \vec{c}) \equiv \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

$$(\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Parallelepiped Volume

Let $\vec{a} =< -1, 2, 5 >$, $\vec{b} =< 2, 2, 7 >$ and $\vec{c} =< 3, 2, -1 >$. Find the oriented volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$. and $\vec{c}$. (The sign tells you whether the right hand rule has been respected.)