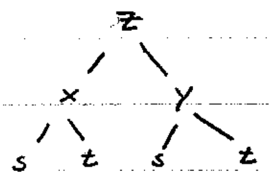
1) Use the chain rule to find $\partial z/\partial s$ and $\partial z/\partial t$

$z = e^{xy} \tan y \qquad x = s+2t, \qquad y = s/t$

Finding our tree diagram



So the chain rule for $\partial z/\partial s$ gives

$$\partial z/\partial s = \partial z/\partial x \; \partial x/\partial s + \partial z/\partial y \cdot \partial y/\partial s$$

$\partial z/\partial x = y e^{xy} \tan y \qquad \partial z/\partial y = x e^{xy} \tan y + e^{xy} \sec^2 y$

$\partial x/\partial s = 1 \qquad \partial y/\partial s = 1/t$

Thus

$$\boxed{\begin{aligned} \partial z/\partial s &= \partial z/\partial x \cdot \partial x/\partial s + \partial z/\partial y \cdot \partial y/\partial s = y e^{xy} \tan y\,(1) + (x e^{xy} \tan y + e^{xy} \sec^2 y)\tfrac{1}{t} \\ &= \tfrac{s}{t} e^{\frac{s}{t}(s+t)} \tan\left(\tfrac{s}{t}\right) + \tfrac{1}{t} e^{\frac{s}{t}(s+t)}\left((s+t)\tan\left(\tfrac{s}{t}\right) + \sec^2\left(\tfrac{s}{t}\right)\right) \end{aligned}}$$

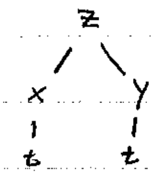
$$\partial z/\partial t = \partial z/\partial x \cdot \partial x/\partial t + \partial z/\partial y \cdot \partial y/\partial t$$

$\partial x/\partial t = 2, \qquad \partial y/\partial t = \frac{\partial}{\partial t} s t^{-1} = -s t^{-2} = -s/t^2$

hence

$$\boxed{\begin{aligned} \partial z/\partial t &= (y e^{xy} \tan(y))(2) + (x e^{xy} \tan y + e^{xy} \sec^2(y))(-s/t^2) \\ &= \tfrac{s}{t} e^{\frac{s}{t}(s+t)}\left(2\tan\left(\tfrac{s}{t}\right) - \tfrac{s+t}{t} \tan\left(\tfrac{s}{t}\right) - \tfrac{1}{t}\left(\sec^2\left(\tfrac{s}{t}\right)\right)\right) \end{aligned}}$$

2) If $z = f(x,y)$, where f is differentiable, $x = g(t)$, $y = h(t)$, $g(3) = 2$, $g'(3) = 5$, $h(3) = 7$, $h'(3) = -4$, $f_x(2,7) = 6$ and $f_y(2,7) = -8$, Find $\partial z/\partial t$ when $t = 3$



The chain rule gives

$$\partial z/\partial t = \partial z/\partial x \cdot \partial x/\partial t + \partial y/\partial x \cdot \partial y/\partial t$$

Now
$$\partial z/\partial x = \partial f/\partial x = f_x(x,y)$$
$$dx/\partial t = dg/\partial t = g'(t)$$
$$\partial z/\partial y = \partial f/\partial y = f_y(x,y)$$
$$dy/\partial t = dh/\partial t = h'(t)$$

so
$$\partial z/\partial t = f_x(x,y)\,g'(t) + f_y(x,y)\,h'(t)$$
$$= f_x(g(t), h(t))\,g'(t) + f_y(g(t), h(t))\,h'(t)$$

from the statement of the problem at $t = 3$

$g(3) = 2$, $h(t) = 7$ and $g'(3) = 5$, $h'(3) = -4$

which implies
$$\partial z/\partial t = f_x(2,7)\cdot(5) + f_y(2,7)(-4)$$

also from the statement of the problem

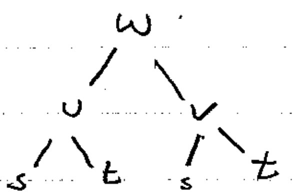
$f_x(2,7) = 6$, $f_y(2,7) = -8$

thus

$$\boxed{\partial z/\partial t = 6 \cdot 5 + (-4)(-8) = 30 + 32 = 62}$$

3) Let $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable, $u(1,0)=2$, $u_s(1,0)=-2$, $u_t(1,0)=6$, $v(1,0)=3$, $v_s(1,0)=5$, $v_t(1,0)=4$, $F_u(2,3)=-1$, and $F_v(2,3)=10$

Find $W_s(1,0)$ and $w_t(1,0)$



Now $\frac{\partial w}{\partial t} = \partial F/\partial u \cdot \partial u/\partial t + \partial F/\partial v \cdot \partial v/\partial t$

$\partial w/\partial s = \partial F/\partial u \cdot \partial u/\partial s + \partial F/\partial v \cdot \partial v/\partial s$

at $(1,0)$ $\quad u(1,0)=2$, $U_t(1,0)=-2$ $\quad U_s(1,0)=-2$

$v(1,0)=3 \quad v_t(1,0)=4 \quad v_s(1,0)=5$

$F_u(u(1,0), v(1,0)) = F_u(2,3) = -1$

$F_v(u(1,0), v(1,0)) = F_v(2,3) = 10$

so $\partial W/\partial t|_{(1,0)} = F_u(2,3)\, u_t(1,0) + F_v(2,3)\, v_t(1,0)$

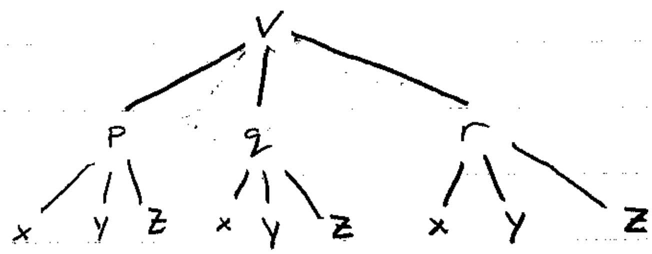
$= (-1)(6) + (10)(4) = -6+40 = \boxed{34}$

and $\partial w/\partial t|_{(1,0)} = F_u(2,3)\, u_s(1,0) + F_v(2,3)\, v_s(1,0)$

$= (-1)(-2) + 10(5) = 2+50 = \boxed{52}$

4 Use a tree diagram to write out the Chain rule for the given case. Assume all functions are differentiable

$v = f(p, q, r)$, $p = p(x, y, z)$, $q = q(x, y, z)$, $r = r(x, y, z)$



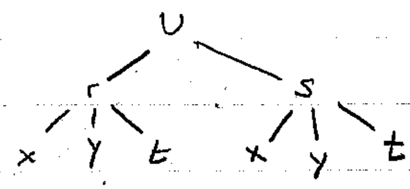
$$\partial v/\partial x = \partial v/\partial p \cdot \partial p/\partial x + \partial v/\partial q \cdot \partial q/\partial x + \partial v/\partial r \cdot \partial r/\partial x$$
$$\partial v/\partial y = \partial v/\partial p \cdot \partial p/\partial y + \partial v/\partial q \cdot \partial q/\partial y + \partial v/\partial r \cdot \partial r/\partial y$$
$$\partial v/\partial z = \partial v/\partial p \cdot \partial p/\partial z + \partial v/\partial q \cdot \partial q/\partial z + \partial v/\partial r \cdot \partial r/\partial z$$

5) Use the chain rule to find the indicated partial derivatives.

$u = \sqrt{r^2+s^2} = (r^2+s^2)^{1/2}$ $\quad r = y + x\cos t \quad s = x + y\sin t$

$\partial u/\partial x, \ \partial u/\partial y, \ \partial u/\partial t$ when $x = 1, y = 2, t = 0$

Things we will need:



$\partial u/\partial r = \frac{1}{2}(r^2+s^2)^{-1/2}(2r) = r(r^2+s^2)^{-1/2}$

$\partial u/\partial s = \frac{1}{2}(r^2+s^2)^{-1/2}(2s) = s(r^2+s^2)^{-1/2}$

$\partial r/\partial x = \cos t, \quad \partial r/\partial y = 1, \quad \partial r/\partial t = -x\sin t$

$\partial s/\partial x = 1, \quad \partial s/\partial y = \sin t, \quad \partial s/\partial t = y\cos t$

Now $\partial u/\partial x = \partial u/\partial r \cdot \partial r/\partial x + \partial u/\partial s \cdot \partial s/\partial x$

$= r(r^2+s^2)^{-1/2}\cos t + s(r^2+s^2)^{-1/2}(1)$

$= \dfrac{(y+x\cos t)(\cos t) + (x+y\sin t)}{((y+x\cos t)^2 + (x+y\sin t)^2)^{1/2}}$

so $\left.\dfrac{\partial u}{\partial x}\right|_{(1,2,0)} = \dfrac{(2+1)(1) + (1+2(0))}{(((2+(1)(1))^2 + (1+2(0))^2)^{1/2}} = \dfrac{4}{\sqrt{9+1}} = \dfrac{4}{\sqrt{10}} = \dfrac{2\sqrt{10}}{5}$

$\partial u/\partial y = \partial u/\partial r \cdot \partial r/\partial y + \partial u/\partial s \cdot \partial s/\partial y$

$= r(r^2+s^2)^{-1/2}(1) + s(r^2+s^2)^{-1/2}\sin t$

$= \dfrac{(y+x\cos t) + (x+y\cos t)\sin t}{((y+x\cos t)^2 + (x+y\sin t)^2)^{1/2}}$

so $\left.\dfrac{\partial u}{\partial y}\right|_{(1,2,0)} = \dfrac{(2+1) + (1+2(0))(0)}{((2+(1)(1))^2 + (1+2(0))^2)^{1/2}} = \dfrac{3}{\sqrt{10}} = \dfrac{3\sqrt{10}}{10}$

$$\partial u/\partial t = \partial u/\partial r \cdot \partial r/\partial t + \partial u/\partial s \cdot \partial s/\partial t$$

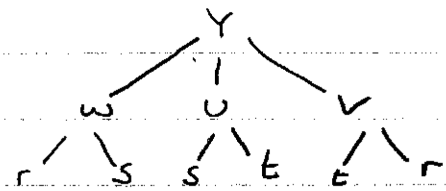
$$= \frac{(y + x\cos t)(-x \sin t) + (x + y \sin t)(y \cos t)}{((y + x \cos t)^2 + (x + y \sin t)^2)^{1/2}}$$

so $$\left.\frac{\partial u}{\partial t}\right|_{(1,2,0)} = \frac{(2+1)(-1(0)) + (1+2(0))(2(1))}{((2+(1)(1))^2 + (1+2(0))^2)^{1/2}} = \frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$$

6) Use the chain rule to find the indicated partial derivatives

$Y = w \tan^{-1}(uv)$, $u = r+s$, $v = s+t$, $w = t+r$

$\partial Y/\partial r$, $\partial Y/\partial s$, $\partial Y/\partial t$ when $r=1$, $s=0$, $t=1$



Some things we will need

$Y_w = \tan^{-1}(uv)$, $Y_u = \frac{w}{1+(uv)^2}$, $Y_v = \frac{w}{1+(uv)^2}$

$\partial u/\partial r = 1$, $\partial u/\partial s = 1$, $\partial u/\partial t = 0$

$\partial v/\partial r = 0$, $\partial v/\partial s = 1$, $\partial v/\partial t = 1$

$\partial w/\partial r = 1$, $\partial w/\partial s = 0$, $\partial w/\partial t = 1$

$u(1,0,1) = 1+0 = 1$

$v(1,0,1) = 0+1 = 1$

$w(1,0,1) = 1+1 = 2$

$Y_w(1,1,2) = \frac{\pi}{4}$ $\quad Y_u(1,1,2) = \frac{2}{1+1} = 1$, $Y_v(1,1,2) = 1$

So $\partial Y/\partial r = \partial Y/\partial w \cdot \partial w/\partial r + \partial Y/\partial u \cdot \partial u/\partial r + \partial Y/\partial v \cdot \partial v/\partial r$

thus $\partial Y/\partial r\,|_{(1,0,1)} = \frac{\pi}{4}(1) + (1)(1) + (1)(0) = \frac{\pi}{4} + 1 = \frac{4+\pi}{4}$

$\partial Y/\partial s = \partial Y/\partial w \cdot \partial w/\partial s + \partial Y/\partial u \cdot \partial u/\partial s + \partial Y/\partial v \cdot \partial v/\partial s$

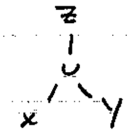
$= \frac{\pi}{4}(0) + (1)(1) + (1)(1) = 2$

$\partial Y/\partial t = \partial Y/\partial w \cdot \partial w/\partial t + \partial Y/\partial u \cdot \partial u/\partial t + \partial Y/\partial v \cdot \partial v/\partial t$

$= \frac{\pi}{4}(1) + (1)(0) + (1)(1) = \frac{\pi}{4} + 1 = \frac{4+\pi}{4}$

7) If $z = f(x-y)$ show $\partial z/\partial x + \partial z/\partial y = 0$

Let $u = x - y$ the



hence $\partial z/\partial x = f_u(x-y) \cdot \partial u/\partial x = f_u(x-y)$

$\partial z/\partial y = f_u(x-y) \cdot \partial u/\partial y = f_u(x-y)(-1) = -f_u(x-y)$

thus

$\partial z/\partial x + \partial z/\partial y = f_u(x-y) - f_u(x-y) = 0$

8) Show that any function of the form
$z = f(x+at) + g(x-at)$ is a solution to the wave equation $\partial^2 z/\partial t^2 = a^2 \partial^2 z/\partial x^2$

proof let $u = x+at$ $v = x-at$

then $z = f(u) + g(v)$

and $\partial z/\partial t = \partial z/\partial u \cdot \partial u/\partial t + \partial z/\partial v \cdot \partial v/\partial t$

Now $\partial z/\partial u = f'(u)$ since $g(v)$ does not depend on u. Similarly $\partial z/\partial v = g'(v)$

Thus $\partial z/\partial t = f'(u)(a) + g'(v) \cdot (-a)$

Let $h(u,v) = \partial z/\partial t$

then $\partial^2 z/\partial t^2 = \partial h/\partial t = \partial h/\partial u \cdot \partial u/\partial t + \partial h/\partial v \cdot \partial v/\partial t$

as above $\partial h/\partial u = af''(u)$ since $(-a)g'(v)$ doesn't depend on u. Also $\partial h/\partial v = -ag''(v)$

Thus $\partial^2 z/\partial t^2 = af''(u)(a) - ag''(v)(-a) =$

$= a^2 f''(u) + a^2 g''(v) = a^2(f''(u) + g''(v))$

Similarly to the above

$\partial z/\partial x = \partial z/\partial u \cdot \partial u/\partial x + \partial z/\partial v \cdot \partial v/\partial x$

this time $\partial u/\partial x = 1 = \partial v/\partial x$

hence $\partial z/\partial x = f'(u) + g'(v)$

similarly Let $l(u,v) = \partial z/\partial x$

then $\partial^2 z/\partial x^2 = \partial l/\partial x = \partial l/\partial u \cdot \partial u/\partial x + \partial l/\partial v \cdot \partial v/\partial x$

$\partial l/\partial u = f''(u)$, $\partial l/\partial v = g''(v)$

hence $\partial^2 z/\partial x^2 = f''(u) + g''(v)$

Thus $a^2 \partial^2 z/\partial x^2 = \partial^2 z/\partial t^2$

9) Find the directional derivative of f at the given point in the direction give by the angle θ

$$f(x,y)=\sqrt{5x-4y} = (5x-4y)^{1/2}, \quad (4,1) \quad \theta=-\pi/6$$

a unit vector in the direction given by the angle θ is just $\langle \cos\theta, \sin\theta\rangle$

$\theta=-\pi/6$ hence our vector is just

$$\langle \cos -\tfrac{\pi}{6}, \sin -\tfrac{\pi}{6}\rangle = \langle \tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\rangle$$

Now from Theorem 3 in this section

$$D_{\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle} f(x,y) = f_x(x,y)(\sqrt{3}/2) + f_y(x,y)(-\tfrac{1}{2})$$

$$f_x(x,y) = \tfrac{1}{2}(5x-4y)^{-1/2}\cdot 5$$
$$f_y(x,y) = \tfrac{1}{2}(5x-4y)^{-1/2}\cdot(-4)$$

hence $D_{\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle} f(4,1) = \left(\tfrac{1}{2}\cdot\tfrac{1}{4}\cdot 5\right)\left(\tfrac{\sqrt{3}}{2}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)(-4)\left(-\tfrac{1}{2}\right)$

$$= \frac{5\sqrt{3}}{16} + \frac{4}{16}$$

10) Find the directional derivative of f at the given point in the direction indicated by the angle θ

$f(x,y) = x\sin(xy)$ $\quad (2,0) \quad \theta = \pi/3$

a unit vector in the direction of $\frac{\pi}{3}$ is $\langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = \langle \frac{1}{2}, \sqrt{3}/2 \rangle$

by theorem 3

$D_{\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle} f(2,0) = f_x(2,0)\frac{1}{2} + f_y(2,0)\frac{\sqrt{3}}{2})$

$f_x(x,y) = \sin(xy) + xy\cos(xy)$
$f_y(x,y) = x^2\cos(xy)$
$f_x(2,0) = \sin(0) + 2(0)\cos(0) = 0$
$f_y(2,0) = 4\cos(0) = 4$

thus $D_{\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle} f(2,0) = 0 \cdot \frac{1}{2} + 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$

11) $f(x,y) = y \ln x$ $P(1,-3)$ $u = \langle -\frac{4}{5}, \frac{3}{5} \rangle$

(a) Find the gradient of f
(b) Evaluate the gradient at the point P
(c) Find the rate of change of f at P in the direction of the vector u.

(a) the gradient of $f = \langle f_x(x,y), f_y(x,y) \rangle$
$f_x(x,y) = \frac{y}{x}$ $f_y(x,y) = \ln x$
so the gradient of f is $\langle \frac{y}{x}, \ln x \rangle$

(b) $\langle \frac{y}{x}, \ln x \rangle \Big|_{(1,-3)} = \langle \frac{-3}{1}, \ln(1) \rangle = \langle -3, 0 \rangle$

(c) we must first find a unit vector in the direction of u.
to do this we take $\frac{u}{|u|}$
Now $|u| = \sqrt{(-\frac{4}{5})^2 + (\frac{3}{5})^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{\frac{25}{25}} = 1$
thus u is a unit vector
and to find the rate of change we take
$\nabla f(1,-3) \cdot u = \langle -3, 0 \rangle \cdot \langle -4/5, \frac{3}{5} \rangle = \frac{12}{5}$

12) $f(x,y,z) = \sqrt{x+yz} = (x+yz)^{1/2}$, $P(1,3,1)$, $u = \langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \rangle$

(a) Find the gradient of f

(b) Evaluate the gradient at the point P

(c) Find the rate of change of f at P in the direction of the vector u

(a) $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$

$f_x = \frac{1}{2}(x+yz)^{-1/2}(1)$

$f_y = \frac{1}{2}(x+yz)^{-1/2}(z)$

$f_z = \frac{1}{2}(x+yz)^{-1/2}(y)$

so $\nabla f = \langle \frac{1}{2\sqrt{x+yz}}, \frac{z}{2\sqrt{x+yz}}, \frac{y}{2\sqrt{x+yz}} \rangle$

(b) $\nabla f(1,3,1) = \langle \frac{1}{2\sqrt{1+3(1)}}, \frac{1}{2\sqrt{4}}, \frac{3}{4} \rangle = \langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \rangle$

(c) $|u| = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = \sqrt{\frac{49}{49}} = 1$

hence u is a unit vector and the rate of change is given by

$\nabla f(1,3,1) \cdot u = \langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \rangle \cdot \langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \rangle$

$= \frac{2}{28} + \frac{3}{28} + \frac{18}{28} = \frac{23}{28}$

13) Find the directional derivative of the function at the given point in the direction v.

$g(s,t) = s^2 e^t \qquad (2,0) \qquad v = i + j$

v is not a unit vector so we first have to find a unit vector in the direction of v. This is just given by $\frac{v}{|v|}$

$|v| = \sqrt{1+1} = \sqrt{2}$ thus $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ is such a vector

$D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} g(2,0) = g_s(2,0)\frac{1}{\sqrt{2}} + g_t(2,0)\frac{1}{\sqrt{2}}$

by theorem 3

$g_s(s,t) = 2se^t \Rightarrow g_s(2,0) = 4$

$g_t(s,t) = s^2 e^t \Rightarrow g_t(2,0) = 4$

hence $D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} g(2,0) = \frac{4}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

14) Find the directional derivative of the function at the given point in the direction of v.

$f(x,y,z) = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$, $(1,2,-2)$, $v = \langle -6, 6, -3 \rangle$

first we will find the unit vector in the direction of v, $|v| = \sqrt{36+36+9} = \sqrt{81} = 9$

hence $\frac{v}{|v|}$ is a unit vector in the direction of v and $\frac{v}{|v|} = \frac{1}{9}\langle -6, 6, 3 \rangle =$

we know from $\boxed{14}$ $D_u f(x,y,z) = \nabla f(x,y,z) \cdot u$ where u is a unit vector

$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$

$f_x = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $f_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$

so $\nabla f(1,2,-2) = \left\langle \frac{1}{\sqrt{1+4+4}}, \frac{2}{3}, \frac{-2}{3} \right\rangle = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$

hence $D_{\frac{v}{|v|}} f(1,2,-2) = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle \cdot \frac{1}{9}\langle -6, 6, -3 \rangle$

$= \frac{1}{9}(-2 + 4 + 2) = \frac{4}{9}$