LECTURE OUTLINE
The Gradient and Extreme Points

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Math 8

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Last Time

Let θ be the angle between $\hat{u}$ and ∇f. Notice, we have

$$D_u f = |\nabla f| \cos(\theta)$$

Consequences: f increases the fastest in the direction of ∇f, f decreases fastest the direction of $-\nabla f$, and f does not change as we head in a direction perpendicular to ∇f.

Contour...

Consequences

Example: Assume we are at the point $(2,1)$. What direction should we head in order to increase $f(x,y)=ye^{x^2}$ at the fastest rate? What direction should we head to follow a level curve?

Level Surfaces

In three dimensions, ∇f gives a vector perpendicular to a *level surface*.

Example: Describe the level surface of the function $f(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2$ corresponding to $f(x, y, z) = 1$. Find the tangent plane of this surface at the point (a, b, c).

Maxima and Minima

A function of two variables has a *local maximum* at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). The number $f(a, b)$ is called a *local maximum value*.

A function of two variables has a *local minimum* at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b). The number $f(a, b)$ is called a *local minimum value*.

If f has a local maximum or minimum at (a, b) and the partial derivatives of f exist there, then $\nabla f = 0$ (the 0 vector).

Critical Points

If $\nabla f(a, b) = 0$, then we call (a, b) a *critical point* of f.

Example: Find the critical points of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Determine whether these points are minima or maxima.

Example: Find the critical points of $f(x, y) = y^2 - x^2$. Determine whether these points are minima or maxima.

Second Derivative Test

Second Derivative Test : Suppose (a, b) is a critical point of $f(x, y)$ and that the second partial derivatives of f are continuous on a disk with center (a, b). Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(a) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local minimum or local maximum. (This is is called a *saddle point*.)

Example

Example 1: Apply the

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

test to our previous examples.

Example 2: Find the critical points of $f(x, y) = x^4 + y^4 - 4xy + 1$. Determine whether these points are minima, maxima, or saddles.