LECTURE OUTLINE
Chain Rule

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Math 8

Nov. 15, 2004

Goals

Chain Rule

Gradient

Tree Diagrams

Review

We can approximate our function $f(x,y)$ with the plane

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

As such, near (a,b) we have $\Delta z = f(x,y) - f(a,b)$ is approximately

$$\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y,$$

and it can be useful to think using the *differential*

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

A Baby Step Towards the General Chain Rule

Recall $dy = \frac{dy}{dx}dx$. From this we have the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

In two dimensions, using our above differential $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ and the same reasoning we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Ex. Let $f(x, y) = x^2y + y^3$, $x(t) = \sin(t)$, and $y(t) = e^t$. Find $\frac{d}{dt}\left(e^t(\sin(t))^2 + e^{3t}\right)$ in the old way and using the chain rule.

The Gradient

Let $\vec{r}(t) =< x(t), y(t) >$ and let

$$\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > .$$

We call ∇f the *gradient* of f. The chain rule becomes

$$\frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt}$$

Example: Let $\vec{r}(3) = (1, -1)$, $\frac{d\vec{r}}{dt}(3) = (1, 2)$, and $\nabla f(1, -1) = (2, 5)$. Compute $\frac{d}{dt}(f(\vec{r}(t))$ at $t = 3$.

The General Chain Rule

Suppose f is a differentiable function of the n variables $x_1, \ldots, x_n$. Let

$$\nabla f =< \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} > .$$

Let $\vec{x} =< x_1, \ldots, x_n >$. Suppose each x_j is a differentiable function of the m variables $t_1, \ldots, t_m$. Then f can be viewed as a function of $t_1, \ldots, t_m$ and

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \vec{x}}{\partial t_i}.$$

Example: Let $\vec{x}(3) = (1, -1, 2)$, $\frac{d\vec{x}}{dt}(3) = (1, 2, 0)$, and $\nabla f(1, -1, 2) = (2, 5, 3)$. Compute $\frac{d}{dt}(f(\vec{x}(t))$ at $t = 3$.

Tree Diagrams and the Chain Rule

The chain rule:

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \vec{x}}{\partial t_i}.$$

Example: Let $f(x, y, z) = z^2 y + y^2 x^2$, $x(t, s) = st$, $y(t, s) = s^2 e^t$, $z(t, s) = t$. Find $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial s}$ in the old way and using the chain rule. Express this using a tree diagram.