Homework due 11/15/04

(1) Two contour maps are shown. One is for a function f whose graph is a cone. The other is for a function g whose graph is a praboloid. Which is which? Why?

I is a paraboloid and II is a cone.
Notice that the circles in II are a constant distance apart and that the ones in I are closer together. This means the height of II is increasing at a constant rate while the height of I is increasing at an increasing rate. Recall that the "slope" along the side of a cone is constant and the "slope" along the side of a paraboloid is changing.

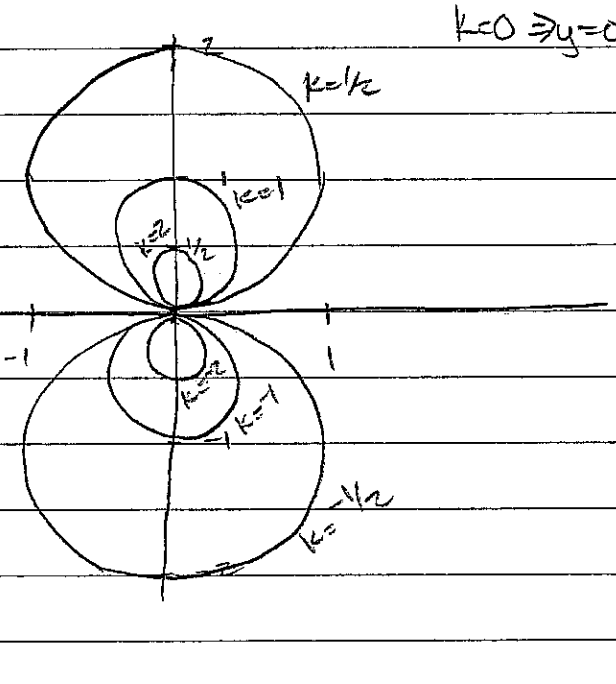
(2) Draw a contour map of the function showing level curves.

$f(x,y) = \dfrac{y}{x^2+y^2}$ Let $k = \dfrac{y}{x^2+y^2} \Rightarrow x^2+y^2 = \dfrac{y}{k}$

$\Rightarrow x^2+y^2-\dfrac{y}{k} = 0 \Rightarrow x^2+\left(y-\dfrac{1}{2k}\right)^2 = \dfrac{1}{4k^2}$

This is a family of circles centered at $(0, \frac{1}{2k})$ with radius $= \frac{1}{4k^2}$

$k=0 \Rightarrow y=0 \Rightarrow$ x axis



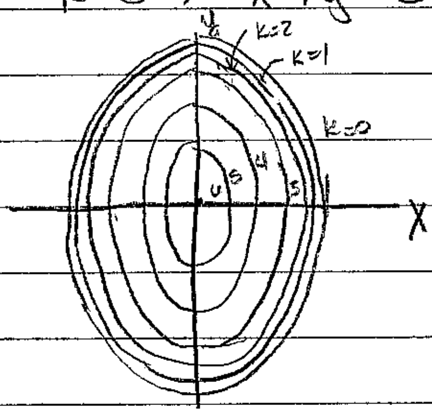
(3) Sketch both the contour map and a graph of the function and compare them.

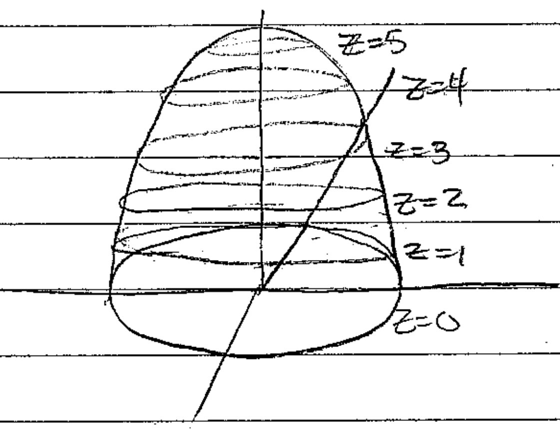
$f(x,y)=\sqrt{36-9x^2-4y^2} \Rightarrow k=\sqrt{36-9x^2-4y^2} \Rightarrow k^2=36-9x^2-4y^2$

$9x^2+4y^2+k^2=36 \Rightarrow \left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2+\left(\frac{k}{6}\right)^2=1$

$\Rightarrow$ a family of ellipses w/ major axis = y axis

$k=6 \Rightarrow x^2+y^2=0 \Rightarrow (x,y)=(0,0)$





If we lift each ellipse on the contour map to a height $z=k$, we have horizontal traces on the graph.

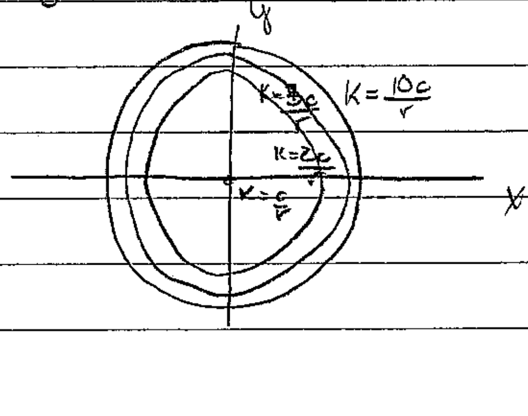
(4) Sketch some equipotential curves if $V(x,y)=\dfrac{c}{\sqrt{r^2-x^2-y^2}}$ $c>0$.

$\Rightarrow k>\frac{c}{\sqrt{r^2}}=\frac{c}{r}$

$k=\dfrac{c}{\sqrt{r^2-x^2-y^2}} \Rightarrow r^2-x^2-y^2=\left(\frac{c}{k}\right)^2 \Rightarrow x^2+y^2=r^2-\left(\frac{c}{k}\right)^2$

$\Rightarrow$ a family of circles. Note: As $k\to\infty$, $\frac{c}{k}\to 0$ $r^2-\left(\frac{c}{k}\right)^2\to r^2$

$k=\frac{c}{r} \Rightarrow x^2+y^2=0 \Rightarrow (x,y)=(0,0)$



(5) Describe the level surfaces of $f(x,y,z) = x^2+3y^2+5z^2$

$K = x^2+3y^2+5z^2 \Rightarrow$ family of ellipsoids, $k>0$

$K=0 \Rightarrow x^2+3y^2+5z^2=0 \Rightarrow (0,0,0)$

(6) Explain why each function is continuous or discontinuous

(a) The outdoor temperature as a function of longitude, latitude & time.

Since small changes in longitude, latitude, & time produce only small changes (no jumps) in temperature, this function is continuous.

(c) The cost of a taxi ride as a function of distance traveled and time.

Most taxis charge a certain amount per fraction of a mile (or per minute) plus a flat fare. So the cost increases in jumps every fraction of a mile (or minute) and so is not continuous

(7 & 8) Find the limit, if it exists, or show that the limit does not exist.

(7) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ First take the limit along the x-axis.

$\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0^2} = 1$. Next, take the limit along the y-axis.

$\lim_{(0,y)\to(0,0)} \frac{0^2}{0^2+y^2} = 0$. Since $1 \neq 0$, the limit does not exist!

(8) $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x^2+y^2)}{(x^2+y^2)} = \lim_{(x,y)\to(0,0)} x^2-y^2 = 0$

(9) Determine the set of points where $F(x,y) = e^{x^2y} + \sqrt{x+y^2}$ is discontinuous.

Both e^{x^2y} & $\sqrt{x+y^2}$ are continuous on their respective domain. i.e. e^{x^2y} is continuous everywhere on $\mathbb{R}^2$ & $\sqrt{x+y^2}$ is continuous when $x+y^2 \geq 0$. So $F(x,y)$ is continuous when $x+y^2 \geq 0$ or $\{(x,y) \in \mathbb{R}^2 \mid x \geq -y^2\}$.

(10) ⓐ What are the meanings of the partial derivatives $\partial h/\partial v$ & $\partial h/\partial t$?

$\partial h/\partial v$ represents the rate of change of h when we fix t. This describes how quickly the wave heights change when the wind speed changes (at a fixed time). $\partial h/\partial t$ represents the rate of change of h when we fix v. This describes how quickly the wave heights change when the time changes (at a fixed wind speed).

ⓑ Estimate the values of $f_v(40,15)$ & $f_t(40,15)$. What are the practical interpretations of these values?

$f_v(40,15) = \lim_{h\to 0} \dfrac{f(40+h,15) - f(40,15)}{h}$ Approximate w/ $h = 10, -10$

$f_v(40,15) \approx \dfrac{f(50,15) - f(40,15)}{10} = \dfrac{36-25}{10} = 1.1$

$f_v(40,15) \approx \dfrac{f(30,15) - f(40,15)}{-10} = \dfrac{16-25}{-10} = .9$

If we take the average, we have $f_v(40,15) \approx 1.0$. This means that when a 40-knot wind has been blowing for 15 hrs, the wave heights should increase by about 1 ft for every knot that the wind increases.

$f_t(40,15) = \lim_{h\to 0} \dfrac{f(40,15+h) - f(40,15)}{h}$ Approximate w/ $h = 5, -5$

$f_t(40,15) \approx \dfrac{f(40,20) - f(40,15)}{5} = \dfrac{28-25}{5} = .6$

$f_t(40,15) \approx \dfrac{f(40,10) - f(40,15)}{-5} = \dfrac{21-25}{-5} = .8$

If we take the average, we have $f_t(40,15) \approx .7$ This means that when a 40 knot wind has been blowing for 15 hrs, the wave heights should increase by .7 feet for extra hour the wind blows

(c) What appears to be the value of $\lim_{t\to\infty} \dfrac{\partial h}{\partial t}$

If we fix v and look at how $f(v,t)$ changes, we see that it increases less & less as t increases, becoming nearly constant. So $\lim_{t\to\infty} \dfrac{\partial h}{\partial t} = 0$.

(11) Use the contour map to estimate $f_x(2,1)$ & $f_y(2,1)$

To estimate $f_x(2,1)$, we start at (2,1) and keep $y=1$ $f(2,1)=10$. If we let x increase, we find $f(2,1)=12$ about .6 units to the right of (2,1). So one estimate of f_x is $2/.6$ If we let x decrease, we find $f(2,1)=8$ about .9 units to the left of (2,1). So another estimate of f_x is $\frac{-2}{-.9}$. If we average these, we find $f_x(2,1) \approx 2.8$. Similarly, to estimate $f_y(2,1)$, we start @ (2,1) and keep $x=1$. If we let y increase, we find $f(2,1)=8$ about .9 units above (2,1). So one estimate of f_y is $\frac{-2}{.9}$. If we let y decrease, we find $f(x,y)=12$ about 1 unit below (2,1). So another estimate for f_y is $\frac{2}{-1}$. Averaging these we have $f_y(2,1) \approx -2.1$.

(12) - (14) Find the first partial derivative of the function.

(12) $f(x,y) = x^5 + 3x^3y^2 + 3xy^4$

$f_x = 5x^4 + 9x^2y^2 + 3y^4$ $\qquad$ $f_y = 6x^3y + 12xy^3$

(13) $f(x,t) = \arctan(x\sqrt{t})$ $\qquad$ $\frac{d}{dy}\arctan(u(y)) = \frac{1}{1+u^2}\frac{du}{dy}$

$f_x = \frac{1}{1+(x\sqrt{t})^2} \cdot \sqrt{t} = \frac{\sqrt{t}}{1+x^2t}$

$f_y = \frac{1}{1+(x\sqrt{t})^2} \cdot x \cdot \frac{1}{2}t^{-1/2} = \frac{x}{2\sqrt{t}(1+x^2t)}$

(14) $f(x,y,z) = x^2e^{yz}$

$f_x = 2xe^{yz}$ $\qquad$ $f_y = x^2e^{yz}z = x^2ze^{yz}$

$f_z = x^2e^{yz} \cdot y = x^2ye^{yz}$

(15) Find all the second partial derivatives of $f(x,y) = \ln(3x+5y)$

$f_x = \frac{1}{3x+5y} \cdot 3 = \frac{3}{3x+5y}$ $\qquad$ $f_y = \frac{1}{3x+5y} \cdot 5 = \frac{5}{3x+5y}$

$f_{xx} = \frac{\partial}{\partial x}(3(3x+5y)^{-1}) = \frac{-3}{(3x+5y)^2} \cdot 3 = \frac{-9}{(3x+5y)^2}$

$f_{xy} = \frac{\partial}{\partial y}(3(3x+5y)^{-1}) = \frac{-3}{(3x+5y)^2} \cdot 5 = \frac{-15}{(3x+5y)^2}$

$f_{yx} = \frac{\partial}{\partial x}(5(3x+5y)^{-1}) = \frac{-5}{(3x+5y)^2} \cdot 3 = \frac{-15}{(3x+5y)^2}$

$f_{yy} = \frac{\partial}{\partial y}(5(3x+5y)^{-1}) = \frac{-5}{(3x+5y)^2} \cdot 5 = \frac{-25}{(3x+5y)^2}$

(16) Find the rate of change of temperature with respect to distance at the point (2,1) in (a) the x-direction and (b) the y-direction.

$$T(x,y) = \frac{60}{1+x^2+y^2} = 60\,(1+x^2+y^2)^{-1}$$

(a) $T_x = \frac{-60}{(1+x^2+y^2)^2} \cdot 2x \qquad T_x(2,1) = \frac{-120(2)}{(1+2^2+1^2)^2} = \frac{-240}{36} = \frac{-20}{3}$

(b) $T_y = \frac{-60}{(1+x^2+y^2)^2} \cdot 2y \qquad T_y(2,1) = \frac{-120(1)}{(1+2^2+1^2)^2} = \frac{-120}{36} = \frac{-10}{3}$

So @ (2,1) the temperature is decreasing at $20/3\ {}^{\circ}C/m$ in the x direction and $10/3\ {}^{\circ}C/m$ in the y-direction.