1. (a.) The general form of the equation of a sphere is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

with center (a,b,c) and radius r.

To match this form, we must complete the square:

$$x^2+y^2+z^2=8x-6z$$

$$x^2-8x+y^2+z^2+6z=0$$

$$(x-4)^2-16+y^2+(z+3)^2-9=0$$

$$(x-4)^2+y^2+(z+3)^2=16+9=5^2$$

so our sphere has center $(4,0,-3)$, radius 5.

(b.) Since $(4-4)^2+1^2+(-3+3)^2=1$, the point $(4,1,-3)$ lies on the sphere with center $(4,0,-3)$, radius 1, and hence inside the sphere with the same center and radius 5.

<u>Alternate answer</u>

Yes. Because <u>any</u> point must lie inside, on, <u>or</u> outside any sphere.

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2. (11 points) Let $\vec{a} =< -2, 4, 2 >$ and $\vec{b} =< 3, 4, 0 >$

(a) Find a vector which has the same direction as $\vec{b}$ but has length 6.

$$|\vec{b}| = \sqrt{3^2+4^2} = 5$$

$$\frac{6}{5}\vec{b} = < \frac{18}{5}, \frac{24}{5}, 0 >$$

(b) Find the cosine of the angle between $\vec{a}$ and $\vec{b}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{10}{\sqrt{24}\,(5)} = \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

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(c) Find the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.

$$\text{area} = |\vec{a} \times \vec{b}|$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 4 & 2 \\ 3 & 4 & 0 \end{vmatrix} = \langle -8, 6, \overset{-20}{\cancel{\ }} \rangle$$

$$= 2\langle -4, 3, -10 \rangle$$

$$|\vec{a} \times \vec{b}| = 2\sqrt{16+9+100} = 2\sqrt{125}$$

$$\text{or } 10\sqrt{5}$$

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3. (9 points) Find the length of the curve given by $\vec{r}(t) =< \cos(t), \sin(t), 2t^{3/2} >$ with $0 \le t \le \frac{1}{3}$.

$\vec{r}\,'(t) = < -\sin t, \cos t, 3t^{1/2} >$

$\int_0^{1/3} |\vec{r}\,'(t)|\, dt = \int_0^{1/3} \sqrt{1+9t}\; dt$

$u = 1+9t \qquad du = 9\, dt$

$\frac{1}{9} \int_1^4 u^{1/2}\, du$

$= \frac{1}{9}\left(\frac{2}{3}\right) u^{3/2} \Big]_1^4$

$= \frac{2}{27}\left[8-1\right] = \frac{14}{27}$

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4. (points 11) Consider the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n2^n}$

(a) Find the radius of convergence of this series.

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)\,2^{n+1}} \frac{n\,2^n}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x-1|}{2} \left(\frac{n}{n+1}\right) = \left|\frac{x-1}{2}\right|$$

Conv. when $\left|\frac{x-1}{2}\right| < 1$, $|x-1| < 2$

$R = 2$

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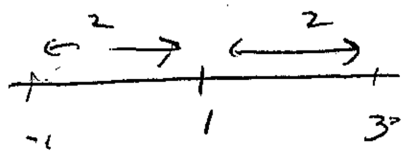
(b) Find the interval of convergence of this series.



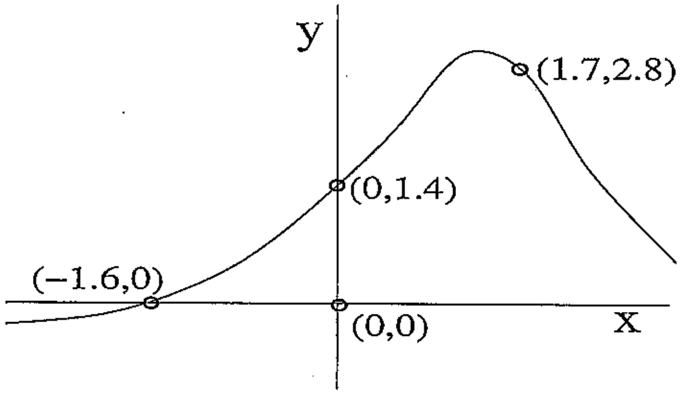
When $x-1=2$, $\sum_{n=1}^{\infty} \frac{2^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$

div. harm series

When $(x-1) = -2$ $\sum_{n=1}^{\infty} \frac{(-2)^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

conv. series (alt. harm)

$-2 \leq |x-1| < 2$



5. (6 points) Using the graph above of $y = f(x)$:

(a) Explain why $.5(x+1.6) - 0.2(x+1.6)^2 + 0.5(x+1.6)^3 + \ldots$ cannot be the Taylor series of $f(x)$ centered at $a = -1.6$.

From the graph, f is concave up at $x = -1.6$, so $f''(-1.6) > 0$. If this were the Taylor series, we'd have

$-0.2 = \frac{f''(-1.6)}{2!}$, so $f''(-1.6)$ would be neg.

(b) Explain why $2.8 + (x-1.7) - 0.5(x-1.7)^2 + 0.4(x-1.7)^3 + \ldots$ cannot be the Taylor series of $f(x)$ centered at $a = 1.7$.

At $x = 1.7$, ~~graph~~ f is decreasing (as can be seen from graph) so $f'(1.7) < 0$.
If this were Taylor series, we'd have $1 = f'(1.7)$, so $f'(1.7)$ would be positive.

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6. (8 points) Find a power series expression for $\int x^2 e^{-x^2}\,dx$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$x^2 e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}$$

$$\int x^2 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+3)\,n!} + C$$

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7. (15 points) Let $f(x) = x\ln(x)$.

(a) Find the second degree Taylor polynomial $T_2(x)$ for f centered at $a = 1$.

$f(1) = 0 \qquad c_0 = 0$

$f'(x) = \frac{x}{x} + \ln x = 1 + \ln x \qquad f'(1) = 1 \qquad c_1 = 1$

$f''(x) = \frac{1}{x} \qquad f''(1) = 1 \qquad c_2 = \frac{1}{2!}$

$(x-1) + \frac{1}{2}(x-1)^2$

(b) Estimate the accuracy of the approximation of $f(x)$ by $T_2(x)$ when x lies in the interval $[.5, 1.5]$. (I.e., give an upper bound for the error.)

$|R_2(x)| \leq \frac{M}{3!} |x-1|^3$

where M is bd. for $|f'''(x)|$ on $[.5, 1.5]$.

$f'''(x) = -\frac{1}{x^2}$

max $|f'''(x)|$ on $[.5, 1.5]$ is $\frac{1}{(.5)^2}$

$= \frac{1}{.25} = \frac{1}{(1/4)} = 4$

$|x-1| \leq .5$ on this interval

$|R_2(x)| \leq \frac{4}{3!}(.5)^3 = \frac{4(.125)}{6} = \frac{.5}{6} = \frac{1}{12}$

8. (a) The parametric equations of the line are

$$x = 2+3t$$
$$y = 1+4t$$
$$z = 5t$$

so two points on the line are

$t=0: \quad (x,y,z) = (2,1,0)$

$t=1: \quad (x,y,z) = (5,5,5)$

so two vectors in the plane are

$$\vec{a} = (5,5,5)-(2,1,0) = (3,4,5)$$
$$\vec{b} = (1,1,1)-(2,1,0) = (-1,0,1)$$

because the point $(1,1,1)$ and the line lie in the plane.

Thus a vector perpendicular to the plane is $\vec{a}\times\vec{b}$.

so let $\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 5 \\ -1 & 0 & 1 \end{vmatrix} = (4,-8,4)$

so the equation of the plane is

$$\vec{n}\cdot(\vec{r}-(1,1,1)) = 0$$
$$(4,-8,4)\cdot(x-1,y-1,z-1) = 0$$
$$4(x-1) - 8(y-1) + 4(z-1) = 0$$
$$4x-8y+4z = 0$$

or $\quad x-2y+z = 0.$

(b.) A line perpendicular to the plane is in the direction of the normal vector of the plane, which is $(5,2,3)$.

So let $\vec{v} = (5,2,3)$.

Then the line has equation

$$\vec{r} = (2,1,4) + t(5,2,3).$$

which in parametric form is

$$x = 2+5t$$
$$y = 1+2t$$
$$z = 4+3t$$

or in symmetric form is

$$\frac{x-2}{5} = \frac{y-1}{2} = \frac{z-4}{3}$$

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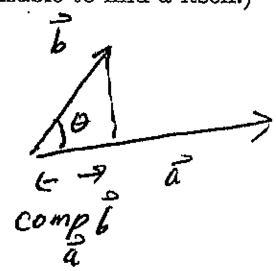
9. (5 points) Let $\vec{b} = \langle 2, 1, 3 \rangle$ and let $\vec{a}$ be another vector in $\mathbf{R}^3$. Suppose that

$$\text{comp}_{\vec{a}}\, \vec{b} = 2,$$

where as usual $\text{comp}_{\vec{a}}\, \vec{b}$ denotes the component (i.e., scalar projection) of $\vec{b}$ along $\vec{a}$. Find the cosine of the angle between $\vec{a}$ and $\vec{b}$. (Note: it is neither necessary nor advisable to find $\vec{a}$ itself.)

Easy way:



$$2 = \text{comp}_{\vec{a}}\, \vec{b} = |\vec{b}| \cos\theta = \sqrt{14} \cos\theta$$

$$\cos\theta = \frac{2}{\sqrt{14}}$$

Another way:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} \qquad \text{comp}_{\vec{a}}\, \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = 2$$

$$\text{So} \quad \cos\theta = \frac{2}{|\vec{b}|} = \frac{2}{\sqrt{14}}$$

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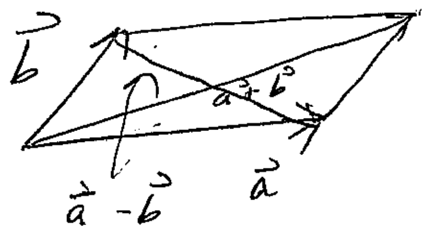
10. (10 points) Consider a parallelogram spanned by two vectors $\vec{a}$ and $\vec{b}$.

(a) One of the diagonals is $\vec{a}+\vec{b}$. Write an analogous expression for the other diagonal.

$\vec{a} - \vec{b}$



(b) Using the expressions in part (a), show that the diagonals are perpendicular if and only if the parallelogram is a rhombus. (Recall: A rhombus is a parallelogram all of whose sides have the same length.)

Diags are perp $\Leftrightarrow$

$(\vec{a}+\vec{b}) \cdot (\vec{a}-\vec{b}) = 0 \quad \Leftrightarrow$

$\vec{a}\cdot\vec{a} - \cancel{\vec{a}\cdot\vec{b}} + \cancel{\vec{b}\cdot\vec{a}} - \vec{b}\cdot\vec{b} = 0$

$\Leftrightarrow \quad \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{b} = 0 \quad \Leftrightarrow$

$|\vec{a}|^2 - |\vec{b}|^2 = 0$

$\Leftrightarrow \quad |\vec{a}| = |\vec{b}|$

$\Leftrightarrow$ sides have same length.