HW for Dec. 1

15.7: 30, 33
15.8: 1, 3, 9, 34

1. First, we find the critical points:

$$f_x(x,y) = 4 - 2x \quad \text{so } f_x(x,y) = 0 \text{ gives } x = 2$$
$$f_y(x,y) = 6 - 2y \quad \text{so } f_y(x,y) = 0 \text{ gives } y = 3$$

$(2,3) \in D = \{(x,y) \mid 0 \leq x \leq 4, 0 \leq y \leq 5\}$ so this is the only critical point in D. $\boxed{f(2,3) = 13}$

Next, we consider the boundary points:

These all lie on one of the lines $x = 0$, $x = 4$, $y = 0$, $y = 5$.

$x = 0$: $f(0,y) = 6y - y^2 = -(y-3)^2 + 9$

This has maximum at $y = 3$. $\boxed{f(0,3) = 9}$

and minimum at $y = 0$ (on D). $\boxed{f(0,0) = 0}$

$x = 4$: $f(4,y) = 6y - y^2 = -(y-3)^2 + 9$

so, again, the maximum on this line is at $(4,3)$. $\boxed{f(4,3) = 9}$

and the minimum on D is at $(4,0)$. $\boxed{f(4,0) = 0}$

$y = 0$: $f(x,0) = 4x - x^2 = -(x-2)^2 + 4$

This has maximum at $x = 2$. $\boxed{f(2,0) = 4}$

and minimum at $x = 0$ or 4. $\boxed{f(0,0) = f(4,0) = 0}$

$y = 5$: $f(x,5) = 4x - x^2 + 5 = -(x-2)^2 + 9$

This has maximum at $x = 2$, $\boxed{f(2,5) = 9}$

and minimum at $x = 0$ or 4 $\boxed{f(0,5) = f(2,5) = 5}$

Thus the absolute maximum of f on D is 13 at $(2,3)$ and the absolute minimum is 0 at $(0,0)$ and $(4,0)$.

2. First find the critical points of f on D:

$$f_x(x,y) = 6x^2$$
$$f_y(x,y) = 4y^3$$

so the only critical point is $(0,0)$. $\boxed{f(0,0) = 0}$

Next consider the boundary. This is the curve $x^2+y^2=1$.

Then $f(x,y) = 2x^3 + (y^2)^2 = 2x^3 + (1-x^2)^2 = x^4+2x^3-2x^2+1$

i.e. we want to maximise and minimise the function of one variable, $g(x) = x^4+2x^3-2x^2+1$, for $-1 \leq x \leq 1$.

$$g'(x) = 4x^3+6x^2-4x = 2x(2x^2+3x-2)$$
$$= 2x(2x-1)(x+2)$$

so the possible critical values on the boundary are at $x=0$, $x=\frac{1}{2}$. [Note $x=-2$ is not in our range.]

$\boxed{g(0)=1}$ $\boxed{g(\tfrac{1}{2}) = \tfrac{13}{16}}$

Lastly we must check the endpoints of the interval $-1 \leq x \leq 1$.

$\boxed{g(-1) = -2}$ $\boxed{g(1) = 2}$

Thus the minimum of f on D occurs at $x=-1$, that is the point $(-1,0)$, where $f(-1,0)=-2$

and the maximum occurs at $(1,0)$, with value 2.

3. Look for the level curves that intersect the line $g(x,y)=8$ of maximum and minimum value.

It should be clear that the minimum is 30 and the maximum is approximately 59.

4. $\nabla f(x,y,z) = (2,6,10)$

$\nabla g(x,y,z) = (2x, 2y, 2z)$. $g(x,y,z) = x^2+y^2+z^2$

so our system of equations is:

$$2 = \lambda \cdot 2x \Rightarrow \lambda \neq 0 \text{ and } x = \tfrac{1}{\lambda} \quad ①$$
$$6 = \lambda \cdot 2y \Rightarrow y = \tfrac{3}{\lambda} \quad ②$$
$$10 = \lambda \cdot 2z \Rightarrow z = \tfrac{5}{\lambda} \quad ③$$
$$x^2+y^2+z^2 = 35 \quad ④$$

Substitute ① ② and ③ into ④: $\left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 + \left(\frac{5}{\lambda}\right)^2 = 35$

so $\lambda^2 = 1$, i.e. $\lambda = \pm 1$.

Replacing this back into ① ② and ③, we see the critical points are $(1,3,5)$ and $(-1,-3,-5)$.

Thus the maximum is $f(1,3,5) = 70$

and the minimum is $f(-1,-3,-5) = -70$.

5. $\nabla f(x,y,z) = (yz, xz, xy)$

$g(x,y,z) = x^2+2y^2+3z^2$ so $\nabla g(x,y,z) = (2x, 4y, 6z)$.

so our system of equations is:

$$yz = 2\lambda x \quad ①$$
$$xz = 4\lambda y \quad ②$$
$$xy = 6\lambda z \quad ③$$
$$x^2+2y^2+3z^2 = 6 \quad ④$$

If any of x, y, z are 0, then $f(x,y,z) = xyz = 0$. So suppose none are 0. Then

$$\left.\begin{aligned} \text{by ①: } \lambda &= \frac{yz}{2x} = \frac{xyz}{2x^2} \\ ②: \lambda &= \frac{xz}{4y} = \frac{xyz}{4y^2} \\ ③: \lambda &= \frac{xy}{6z} = \frac{xyz}{6z^2} \end{aligned}\right\} \text{these imply } 2x^2 = 4y^2 = 6z^2 \text{ or } x^2 = 2y^2 = 3z^2.$$

Substitute this into ④: $x^2+x^2+x^2 = 6$

so $x^2 = 2$

$x = \pm\sqrt{2}$

so the critical points are $(\pm\sqrt{2}, \pm 1, \pm\sqrt{2/3})$

$f(\pm\sqrt{2}, \pm 1, \pm\sqrt{2/3}) = \pm \frac{2}{\sqrt{3}}$.

thus the maximum value of f is $2/\sqrt{3}$

and the minimum is $-2/\sqrt{3}$.

6. Our problem is to maximise $f(x,y,z) = xyz$

subject to $2xy+2xz+2yz=64$

Let. $g(x,y,z) = xy+xz+yz$.

Our system of equations is:

$$yz = \lambda(y+z) \quad ①$$
$$xz = \lambda(x+z) \quad ②$$
$$xy = \lambda(x+y) \quad ③$$

$xy+xz+yz = 32 \quad ④$

There are many ways in which one might attempt to solve this. Here is one:

From ①, $yz \cdot x = \lambda(y+z) \cdot x = \lambda xy + \lambda xz$

From ②, $xz \cdot y = \lambda(x+z) \cdot y = \lambda xy + \lambda yz$.

Subtracting the above two, $0 = \lambda xz - \lambda yz = \lambda z(x-y)$. ⑤

If $\lambda = 0$, then by ① $yz=0$, so $xyz=0$. This is not the maximum volume possible, so we must have $\lambda \neq 0$. Similarly $z \neq 0$, so by ⑤ we must have $x-y=0$, i.e. $x=y$.

By symmetry of the equations ① ② and ③, we may apply the same argument just done to conclude $x=z$.

Thus $x=y=z$. Substitute this into ④: $3x^2=32$

so $x=y=z=\sqrt{\frac{32}{3}} = \frac{4\sqrt{2}}{\sqrt{3}}$ (side lengths cannot be negative)

So the maximum volume is $xyz = \left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^3 = \frac{128\sqrt{2}}{3\sqrt{3}}$.