1a Evaluate: $\int 9\sqrt{x}\,\ln(x)\,dx$

We shall use integration by parts.

Let $u = \ln x \qquad dv = 9\sqrt{x}\,dx$

$du = \frac{dx}{x} \qquad v = 9\left(\frac{2}{3}\right)x^{3/2} = 6x^{3/2}$

$\int u\,dv = uv - \int v\,du$

$$\int 9\sqrt{x}\ln(x)\,dx = (\ln x)(6x^{3/2}) - \int (6x^{3/2})\frac{dx}{x}$$

$$= 6x^{3/2}\ln x - \int 6x^{1/2}\,dx$$

$$= 6x^{3/2}\ln x - 6\left(\frac{2}{3}\right)x^{3/2} + C$$

$$= 6x^{3/2}\ln x - 4x^{3/2} + C$$

1 b Evaluate: $\int \frac{dx}{(4-x^2)^{3/2}}$

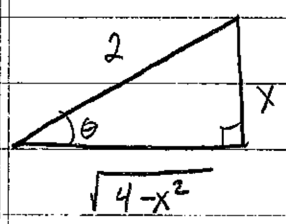
We shall use trig substitution.

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$ So $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\sqrt{1-\sin^2\theta} = 2\cos\theta$

$$\therefore \int \frac{dx}{(4-x^2)^{3/2}} = \int \frac{2\cos\theta}{(2\cos\theta)^{3}}\, d\theta = \int \frac{d\theta}{4\cos^2\theta}$$

$$= \frac{1}{4}\int \sec^2\theta\, d\theta = \frac{1}{4}\tan\theta + C$$



So, $\frac{1}{4}\tan\theta + C = \frac{1}{4}\left(\frac{x}{\sqrt{4-x^2}}\right) + C$

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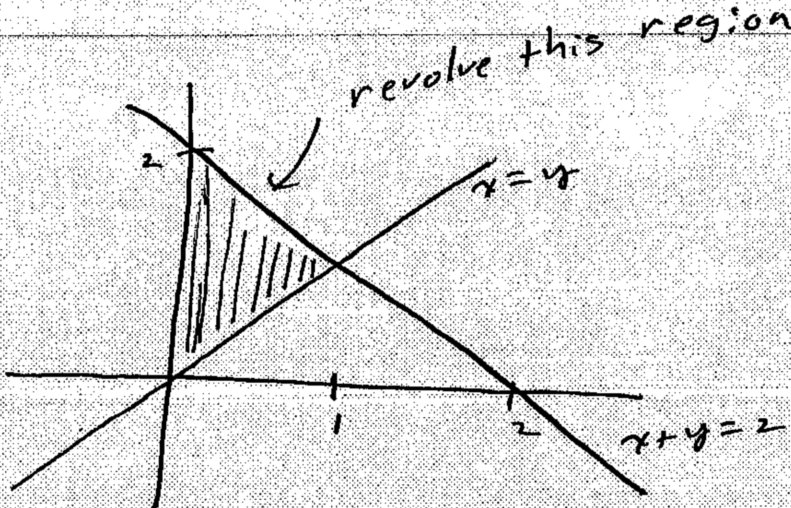
3. (10 points) Write down a definite integral which expresses the volume generated when the region bounded by the lines $x + y = 2$, $x = y$, and the y-axis is revolved about the x-axis.



region bded above by $y = 2 - x$
below by $y = x$

volume generated =

$$\pi \int_0^1 (2-x)^2\,dx - \pi \int_0^1 x^2\,dx$$

$$= \pi \int_0^1 \left[(2-x)^2 - x^2\right] dx$$

4) Find the sum of the following series or show that it converges

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{n-1}}{5^{n+1}}$$

This is a geometric series, we need to get it in the form $\sum_{n=0}^{\infty} ar^n$ to find the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{n-1}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{5^2\, 5^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{25}\left(\frac{(-2)^n}{5^n}\right) = \sum_{n=0}^{\infty} \frac{1}{25}\left(\frac{-2}{5}\right)^n$$

We now have it in the proper form and can see $a = \frac{1}{25}$, $r = -\frac{2}{5}$

clearly $|r| = \frac{2}{5} < 1$ which implies the series converges

and $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Hence $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2)^{n-1}}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{25}\left(\frac{-2}{5}\right)^n = \frac{\frac{1}{25}}{1-\left(-\frac{2}{5}\right)} = \frac{\frac{1}{25}}{\frac{7}{5}}$$

$$= \frac{5}{7(25)} = \frac{1}{35}$$

5 Determine whether the following integral converges

$$\int_1^\infty \frac{2-\sin(x)}{\sqrt{x}}\,dx$$

$$\frac{2-\sin(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} \geq \frac{1}{x} \quad (\text{for } x \geq 1)$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{2-\sin(x)}{\sqrt{x}}\,dx$ diverges

and $\int_1^\infty \frac{2-\sin(x)}{\sqrt{x}}\,dx$ diverges as well.

6) Determine whether the following series conveges absolutely conditionally or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

First we will check absolute convergence

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n \ln(n)}{n}\right| = \sum_{n=1}^{\infty} \frac{\ln n}{n}$$

now $\ln n > 1$ for $n \geq 3$

thus $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$

and we know from the p-series test $(p=1)$ that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

hence by the comparison test $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent

Thus $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ is not absolutely convergent.

Conditional convergence?

we will use the alternating series test

$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ is clearly alternating with $b_n = \frac{\ln n}{n}$

is it decreasing

Let $f(x) = \frac{\ln x}{x}$ if $\frac{\partial}{\partial x} f(x) < 0$ then $\frac{\ln n}{n}$ is decreasing

$$\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x}\left(\frac{\ln x}{x}\right) = \frac{\partial}{\partial x}\left(x^{-1} \ln x\right) = -x^{-2} \ln(x) + x^{-1} \cdot x^{-1}$$

$$= -\frac{\ln(x)}{x^2} + \frac{1}{x^2} = \frac{1 - \ln(x)}{x^2} < 0$$

as long as $\ln(x) > 1$

$\ln(x) > 1$ when $x > e$ thus b_n is decreasing for $n \geq 3$.

$\lim\limits_{n\to\infty} b_n = 0$

Let $f(x) = \frac{\ln x}{x}$ if $\lim\limits_{x\to\infty} f(x) = 0$ then $\lim\limits_{n\to\infty} b_n = 0$

now $\frac{\lim\limits_{x\to\infty} \ln(x)}{\lim\limits_{x\to\infty} x} = \frac{\infty}{\infty}$ is an indeterminate form thus we can apply L'hopital's rule

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{\frac{\partial}{\partial x}\ln(x)}{\frac{\partial}{\partial x} x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Thus $\lim\limits_{n\to\infty} b_n = 0$

Hence by the alternating series test

$\sum\limits_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges

Hence it is conditionally convergent

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7 (b) $\sum_{n=1}^{\infty} \cos(\frac{1}{n})$

$a_n = \cos(\frac{1}{n})$

$\lim_{n \to \infty} \cos(\frac{1}{n}) = \cos(0) = 1$

Since $\lim_{n \to \infty} a_n \neq 0$, the series diverges by (Test for Divergence)